General direct

THE AMERICAN MATHEMATICAL MONTHLY

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kühn, Ohio State University, Columbus, Ohio.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The following sixteen persons and four institutions have been elected to membership in the Association, on applications duly certified:

To Individual Membership

W. E. BYRNE, Ph. D. (Rennsselaer). Barboursville, W. Va.

M. G. CARMAN, Py.D. (Illinois). Head of Dept., Murray State Teachers Coll., Murray, Ky.

NELSON DUNFORD, Asst., Washington Univ., St. Louis, Mo. (Endorsed by Professors Roever and Dunkel).

MARY B. FERFUSON, A.B. (Waynesburg). Instr., Waynesburg Coll., Rogersville, Pa.

K. W. FOLLEY, Ph.D. (Toronto). Asst. Prof., Trinity Coll., Hartford, Conn.

W. W. GARNETT, Asst., Univ. of Kentucky, Lexington, Ky. (Endorsed by Professors Davis and Latimer).

DAN HALL, A.M. (North Carolina). Instr., A. and M. Coll. of Texas, College Station, Texas.

MARGARET E. JONES, A.M. (Ohio State). Instr., Ohio State Univ., Columbus, Ohio.

C. A. KELLER, A.M. (California). Instr. Lehigh Univ., Bethlehem, Pa.

A. S. MCMASTER, C.E. (Texas), M.S. in C.E. (Colorado). Instr., Engg. Math., Univ. of Colorado, Boulder, Colo.

D. W. QUERFELD, A.B. (James Millikin). Instr. High School, Louisville, Ill.

B. D. ROBERTS, A.M. (Indiana). Prof., Parsons Coll., Fairfield. Iowa.

L. J. SCHNELL, B.S. in E.E. (Colorado). Instr., and Head of Dept., Augsburg Coll., Minneapolis, Minn.

M. B. TOLAR, A.M. (Wake Forest; Kentucky). Head of Dept., Shurtleff Coll., Alton Ill.

C. C. WAGNER, A. M. (Penn. State). Asso. Prof., Penn. State Coll., State College, Pa.

OSCAR ZARISKI, Ph.D. (Rome). Holder of the Johnston Scholarship, 1927–28, 1928–29, Johns Hopkins Univ., Baltimore, Md.

To Institutional Membership

AUGSBURG COLLEGE, Minneapolis, Minn.

UNIVERSITY OF MISSISSIPPI, Oxford, Miss.

STATE TEACHERS COLLEUE, Peru, Nebr.

WESTMINSTER COLLEGE, New Wilmington, Pa.

THE NOVEMBER MEETING OF THE SOUTHERN CALIFORNIA SECTION

The eighth regular meeting of the Southern California Section was held at Fullerton Junior College, Fullerton, California, on Saturday, November 3, 1928. Professor E. T. Bell presided.

The attendance was forty-eight, including the following thirty-one members of the Association: O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, Harry Bateman, Clifford Bell, E. T. Bell, Grace E. Berry, W. N. Birchby, Jessie R. Campbell, P. H. Daus, Iva B. Ernsberger, Raymond Garver, Harriet E. Glazier, W. L. Hart, E. R. Hedrick, G. H. Hunt, W. E. Mason, A. W. Prater, Lena E. Reynolds, W. P. Russell, G. E. F. Sherwood, H. M. Showman, D. V. Steed, Morgan Ward, L. E. Wear, Mabel G. Whiting, W. M. Whyburn, Clyde Wolfe, Euphemia R. Worthington.

The following program was presented:

1. "A note on the evaluation of two definite integrals," by Professor H. Bateman, California Institute of Technology.

2. "An application of a Tschirnhaus transformation to elementary theory of equations," by Professor Raymond Garver, University of California at Los Angeles.

3. "A complex space," by Professor L. E. Wear, California Institute of Technology.

4. "Boundary value problems for second order differential systems," by Professor W. M. Whyburn, University of California at Los Angeles.

5. "Relations satisfied by the binomial series," by Dr. Morgan Ward, California Institute of Technology.

6. "Conservative new-type examinations for college mathematics and their coefficient of reliability," by Professor W. L. Hart, University of Minnesota.

Abstracts of these papers follow:

1. The well known inversion formula of Fourier may be generalized to meet the case in which the product $\cos(xt)\cdot\cosh(xt)$ occurs in the first integral instead of $\cos(xt)$. The inversion formula is of a slightly different type and is applicable only to a certain class of functions. A definite integral mentioned in a previous verification of the inversion formula in a special case is now evaluated rigorously. A similar integral is also evaluated to check the corresponding generalization of Fourier's sin-formula.

2. By applying a simple Tschirnhaus transformation to the roots of the reduced cubic, the cubic is transformed to the form $z^3+Az^2-\Delta=0$, where Δ is the discriminant of the cubic. In this form, the relation between the nature of the roots and the discriminant is readily illustrated graphically.

3. A point in three-dimensional space may be determined by a complex variable z and a real variable t. An equation connecting z and t will be a space curve. Certain equations are discussed and the map equations of some surfaces given.

4. In an article that is to appear in the Transactions of the American Mathematical Society, the author introduces a transformation that when applied to a class of non-linear differential equations of the second order yields existence and oscillation theorems for these equations and boundary conditions of the Sturmian type. The class of non-linear systems treated includes the linear system, and for this case the methods of the paper yield a very simple derivation of many of the properties that were established by Sturm, Bôcher, and others. The present paper is confined primarily to the linear case of the above article.

5. Some functional equations are considered and some of their properties developed. The connection between these functions and the binomial series is explained.

6. Professor Hart argues against the use of the typical varieties of questions found in so-called objective examinations as the sole foundation for a reformed system of examinations in mathematical examinations. He advises the use of an alternative variety of short-answer questions of a direct nature, as contrasted with the indirectness of the typical objective examination. For a particular new-type examination of this conservative type which Professor Hart gave to six hundred students, the coefficient of reliability is approximately .9.

P. H. DAUS, *Secretary.*

GRAPHICAL INTEGRATION AND DIFFERENTIATION OF FUNCTIONS IN A POLAR CÖORDINATE SYSTEM

By E. A. KHOLODOVSKY, New York, N. Y.

1. In his book on *Graphical Methods*[1] (Columbia University lectures) Carl Runge gives methods of graphical integration of functions in a Cartesian coördinate system, that is, the methods of finding graphically the curve whose equation in Cartesian coördinates could be written

$$y = \int_{x_0}^{x} f(x)dx,$$

when the curve $y=f(x)$ is given graphically, and also methods of graphical differentiation.

In some cases the solution of a problem requires graphical integration and differentiation when the equation of the curve, which is given graphically, would be written in polar coördinates. The methods of this graphical integration and differentiation in a polar coördinate system will be expounded in this article.

[1] For the literature of the subject see: *Encyklopädie der Mathematischen Wissenschaften* B. II$_3$. N2 (or the French edition); C. Runge, *Graphische Methoden* (1914); B. Mehmke, *Leitfaden zum graphischen Rechnen* (1924); J. Massau, *Mémoire sur l'intégration graphique* (1885); Fr. Willers, *Graphische Integration* (1920); D'Ocagne, *Calcul graphique et nomographie* (1908).

GRAPHICAL INTEGRATION

2. *The problem of graphical integration.* The problem of integration can be formulated as follows. Let $\rho=\phi(\theta)$ be the polar equation of the given curve AB (or a broken line) with the point 0 as the pole. It is required to find the curve of variation of the area of a sector bounded by the given curve and two vectors from the point O when the polar angle θ increases (Fig. 1); in other terms to find the function

$$\Phi(\theta) = \frac{1}{2}\int_{\theta_0}^{\theta} \phi^2(\theta)d\theta.$$

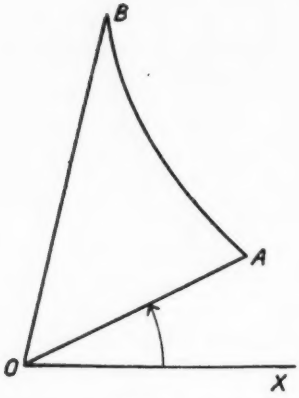

FIG. 1.

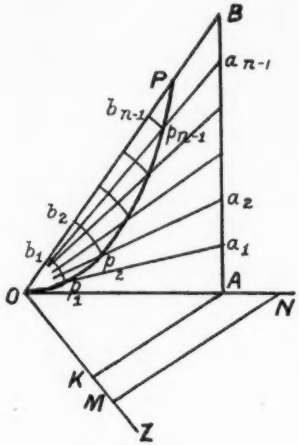

FIG. 2.

The curve may be closed with the point O inside the curve (Fig. 15) or outside (Fig. 9).

If the function is given by an equation $\rho=\phi(\theta)$ we can graph the curve representing the function and then apply the graphical method of integration.

3. *Area of a triangle with a constant base.* Let us begin with a simple particular case of finding the curve of variation of the area of a right triangle OAB (Fig. 2) with a constant base $OA=a$ and increasing altitude AB. On an arbitrary axis OZ put the segment OK equal to the unit of length and the segment $OM=\frac{1}{2}AB$. Join the points A and K and draw MN parallel to KA, N being the point of intersection of this line with the line OA. On the side OB we lay the length of the segment ON; that is we make $OP=ON$. The number measuring the length of the segment OP gives the measure of the area of the triangle OAB in quadratic measure, for $ON/OM=OA/OK$, $ON=\frac{1}{2}AB\cdot OA$. If we divide AB in n equal parts and join the points of division $a_1, a_2, \cdots, a_{n-1}$ with the point O, we obtain the triangles OAa_1, $Oa_1a_2, \cdots, Oa_{n-1}b$ with the area of each equal to OP/n. Dividing OP in n equal parts Ob_1, $b_1b_2, \cdots,$

$b_{n-1}P$ and drawing circles with the center at O through the points $b_1, b_2, \cdots, b_{n-1}$, we get the points $p_1, p_2, \cdots, p_{n-1}$, of intersection of these circles with corresponding rays $Oa_1, Oa_2, \cdots, Oa_{n-1}$. The vectors $Op_1, Op_2, \cdots, Op_{n-1}$ measure correspondingly the areas of the triangles $OAa_1, OAa_2, \cdots, OAa_{n-1}$, and thus we get the points $O, p_1, p_2, \cdots, p_{n-1}$, of the integral curve.

If we take the point O as the pole and OA as the polar axis, the polar equation of AB is $\rho = a/\cos\theta$ $(a = OA)$, and the equation of the obtained integral curve is

$$\Phi(\theta) = \frac{1}{2}\int_0^\theta \frac{a^2 d\theta}{\cos^2\theta} = \frac{a^2}{2}\tan\theta.$$

4. *When the given curve is represented by a straight line.* We shall find the integral curve representing the variation of the area bounded by an arbitrary straight line AB and vectors OA and OB, when the vector OB rotates from coincidence with the vector OA to the position OB (Fig. 3).

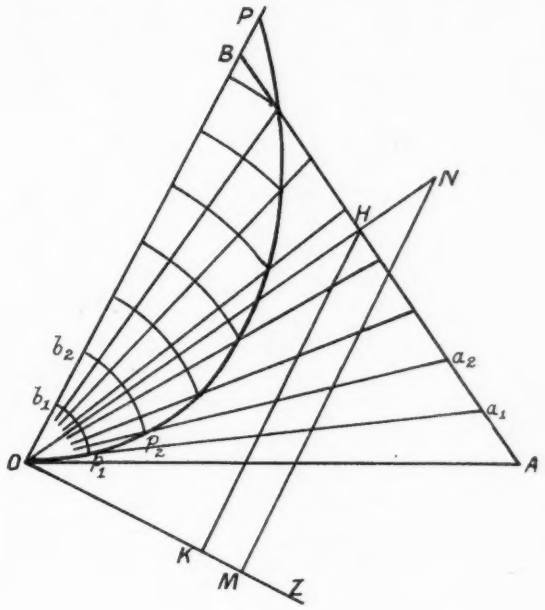

FIG. 3.

Construction: (1) Draw the perpendicular OH to AB, point H being the foot of this perpendicular. (2) On an arbitrary axis OZ construct $OK = 1$ and $OM = \frac{1}{2}AB$. (3) Join the points K and H and draw through the point M a parallel to KH. The point N is the intersection of this line with the perpendicular OH. (4) On OB we lay the length $OP = ON$. The length of the vector OP is the measure of the area of the triangle OAB (as in §3).

Dividing AB and OP in n equal parts we obtain the integral curve in the same way as in §3.

5. *The direction of the axis OZ.* The direction of the axis OZ on which we construct the unit-segment OK is arbitrary, but we shall obtain the most exact sketch if the angle ONM and consequently the angle OHK is greatest within the limits 0 and $\pi/2$. This we obtain when $HK \perp OH$ in case $OH<1$ and when $HK \perp OZ$ in case $OH<1$ (Fig. 3). The maximum of

$$\tan OHK = (\sin HOK)/(h - \cos HOK)$$

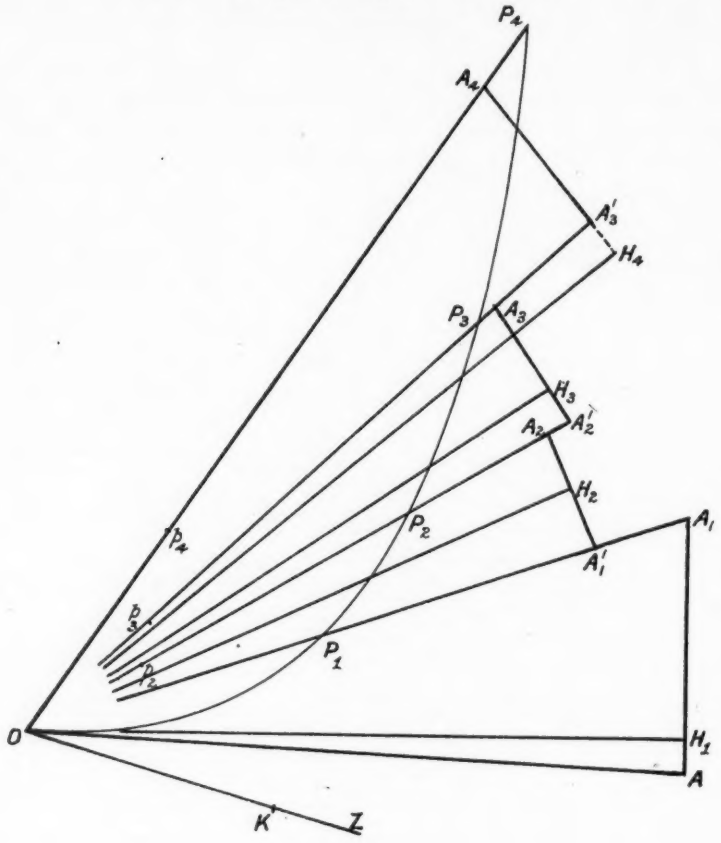

FIG. 4.

occurs when $\cos HOK = h$ in first case and when $\cos HOK = 1/h$ in second case. $h = OH$. Therefore the best position for the axis OZ is one making approximately a right angle with the direction HK in case $OH>1$.

6. *When the given curve is represented by a broken line.* We shall demonstrate the methods of finding the integral curve in case the area is bounded by a broken line and vectors issued from the point O.

First method (Fig. 4): By joining the vertices of the broken line $AA_1A_1'A_2A_2'A_3A_3'A_4$ with the point O we divide the given area into triangles and we can find the area of each triangle separately as in §4. In this way we obtain on the rays OA_1, OA_2, OA_3, OA_4 the vectors OP_1, Op_2, Op_3, Op_4 measuring correspondingly the areas of the triangles OAA_1, $OA_1'A_2$, $OA_2'A_3$, $OA_3'A_4$. Adding to the vector Op_2 on the ray OA_2 the length OP_1, $Op_2+OP_1=OP_2$; adding to the

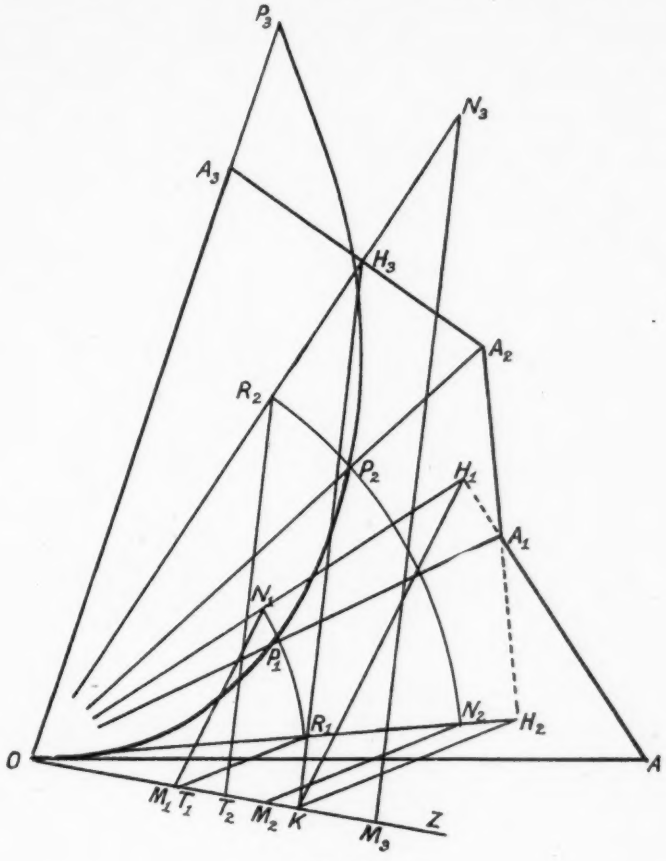

FIG. 5.

vector Op_3 on the ray OA_3 the length OP_2, $Op_3+OP_2=OP_3$; and so on; we obtain the points P_1, P_2, P_3, $\cdots$, of the integral curve.

Following the method of §3 and §4 we can obtain, if necessary, the intermediate points of the curve. We take for different triangles different axes OZ to obtain most accurate results.

Second method (Fig. 5). Instead of constructing vectors measuring the area of each triangle separately it is possible to obtain directly the vectors $OP_1, OP_2, \cdots$, which measure the areas $OAA_1, OAA_1A_1'A_2, \cdots$.

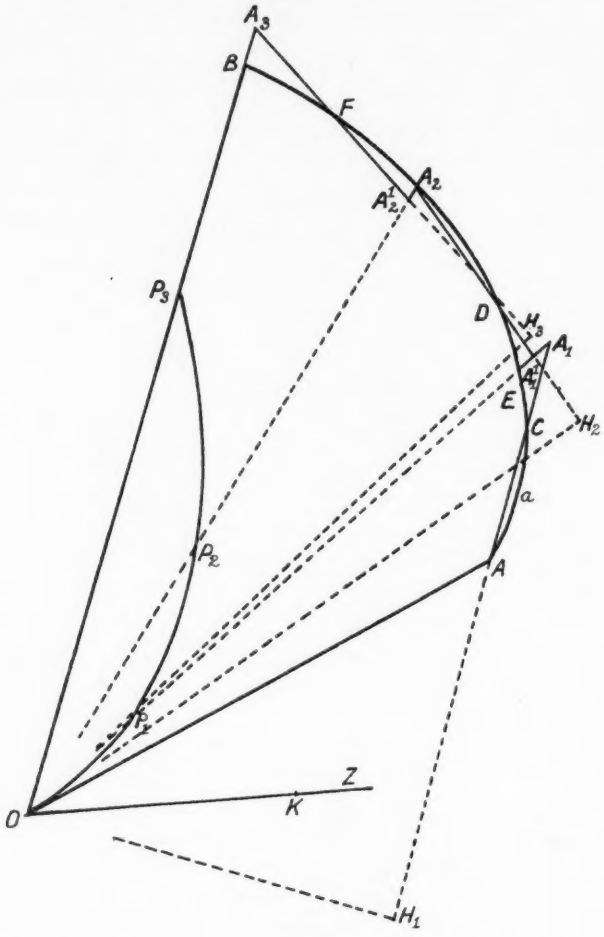

FIG. 6.

Construction: (1) Find the vector OP_1 measuring the area of the triangle OAA_1 as before. (2) On the line OH_2 perpendicular to A_1A_2 lay the length $OR_1 = OP_1$. (3) Join the points K and H_2 and draw the line $R_1T_1 \| KH_2$. (4) From the point T_1 lay off the length $T_1M_2 = \frac{1}{2}A_1A_2$. (5) Draw M_2N_2 parallel to KH_2, N_2 being the intersection of this line with the perpendicular OH_2. (6) On the line OA_2 construct the segment $OP_2 = ON_2$.

The measure of the vector OP_2 is the measure of the area OAA_1A_2, for

$$ON_2/OH_2 = OM_2/OK\ ;\quad ON_2 = OH_2(T_1M_2 + OT_1) = OH_2(A_1A_2/2) + OH_2 \cdot OT_1.$$

But $OH_2/OR_1 = OK/OT_1$, $OH_2 \cdot OT_1 = OR_1 = OP_1$, and $OP_2 = ON_2 = OH_2\ (A_1A_2/2) + OP_1$, the area of the triangles OA_1A_2 and OAA_1.

Proceeding in the same way, on OH_3 we construct $OR_2 = OP_2$; then we draw $R_2T_2 \| KH_3$ and lay off $T_2M_3 = (A_2A_3/2)$. Next we draw $M_3N_3 \| KH_3$ and construct $OP_3 = ON_3$, and so on.

7. *General case.* If the area is bounded by a curve AB and vectors OA and OB (Fig. 6) we replace the given curve by a broken line $AA_1A_1'A_2A_2'A_3$ as in integration in a Cartesian coördinate system, so that the area which is added is compensated by an area which is subtracted. Thus we subtract the area AaC and add the area CA_1E, etc.

The integral curve of this broken line gives us an approximate integral curve for the given curve which may be corrected taking in consideration added and subtracted areas.

With a careful drawing we can obtain a very close approximation. When drawing, it is more convenient to construct first the broken line and, joining all vertices $A_1, A_2, \cdots$ with O, then to draw all perpendiculars $OH_1, OH_2, \cdots$ and bisect all lines $AA_1, A_1'A_2, \cdots$, and afterwards to proceed to locate the points of the integral curve with the OZ axis suitably chosen.

The radii vectors OP_1, OP_2, OP_3, (Fig. 6) of the integral curve corresponding to the points E, A_2, B, of the given curve, where the added and subtracted areas are compensated, represent the exact value of the required sectorial area as nearly as the added areas equal the corresponding subtracted ones.

If we are not satisfied with estimating the equivalence of these areas by eye, we may draw mechanically a broken stepping line to take place of the given curve applying the same construction as in the case of a Cartesian system. (See any of the books referred to at the beginning of this article.)

In Fig. 7 the stepping line $A_1A_2 \cdots A_8$ replaces the given curve $Aabcde\,A_8$. The area A_1A_2a equals the area aA_3b, the area bA_4c equals the area cA_5d, the area dA_6e, the area eA_7A_8. At the points A_1, b, d, A_8 of the given curve, the corresponding radii vectors of the integral curve represent the exact value (in the limits of exactitude of the drawing) of the required sectorial area.

The stepping line has the advantage that the altitudes of the triangles all coincide with the straight lines OA_1, or OB perpendicular to OA_1, though it increases the number of segments required for construction. We may diminish the number of segments without decreasing the accuracy of substituted areas as shown in Fig. 8: $A_2C_1 = C_1A_3$, $A_4C_2 = C_2A_5$, $A_7C_3 = C_3A_8$. Instead of 8 triangles we have only 5, substituting the broken line $A_1B_1B_2A_6B_3A_9$ for the stepping line.

8. *A closed curve.* If it is necessary to find the area of a closed curve $ABCD$ (Fig. 9), we draw the tangents OA and OC to the curve and we construct the

integral curves for the curves ABC and CDA. The difference of the vectors for the same angle will give the measure of the corresponding part of the area bounded by the closed curve and corresponding radius vector. So Op is the measure of the area ABD.

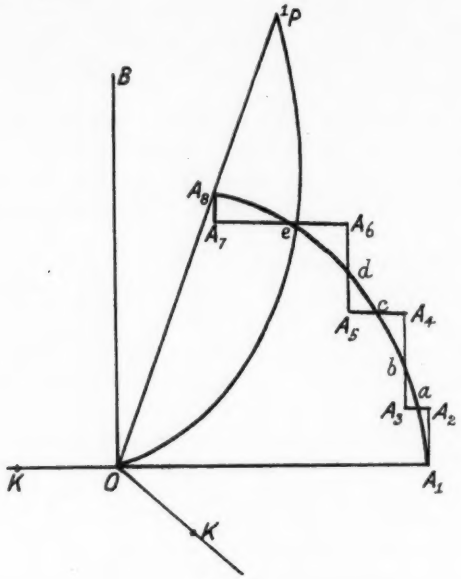

FIG. 7.

9. *A particular case.* We proceed in the same manner, if the vector intersects the given curve in several points (Fig. 10). Drawing the tangents OC and OB we find by subtraction the area CBD. When this area is subtracted from the area OAB we get the required area.

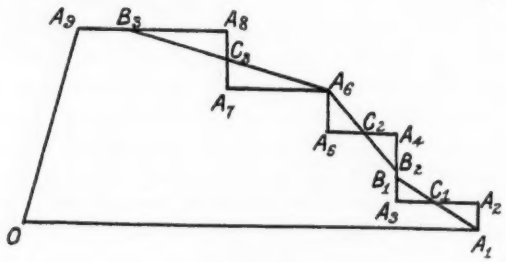

FIG. 8.

10. *Adding of a constant area.* Usually the integral curve in beginning is very flat which makes it difficult to define the points of the given curve corresponding to different radii vectors of the integral curve. We can avoid this

difficulty by adding an arbitrary constant area. This is accomplished by adding to each radius vector (constructed as shown previously) the same length ρ_0. The points obtained belong to the curve whose equation is $\rho = \Phi(\theta) + \rho_0$.

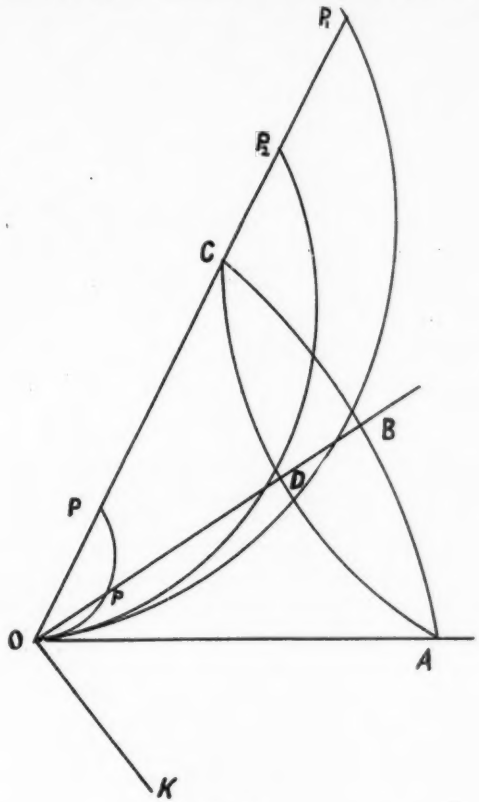

FIG. 9.

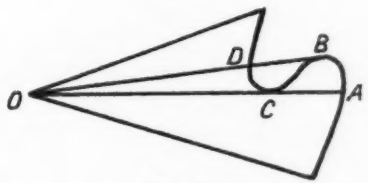

FIG. 10.

We may proceed in another manner: we put on the initial ray OA (Fig. 11) the length $OP_0 = \rho_0$ measuring this added constant area and then proceed as shown in §6. On OH_1 lay the length $OR_0 = OP_0$; join K and H_1; draw R_0T_0 parallel to KH_1; lay $T_0M_1 = \frac{1}{2}AA_1$; draw M_1N_1 parallel to KH_1; and put $OP_1 =$

ON_1. It is easy to see that the vector OP_1 measures the area of the triangle OAA_1 plus the area measured by the vector OR_0. $OP_1 = p_1A_1 + Op_1$. Then we continue as in §6. Every radius vector of the integral curve is increased by the same constant.

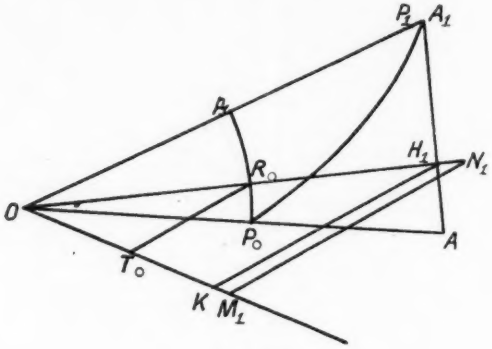

FIG. 11.

11. *Division of the area of a sector in equal sectors.* By use of graphical integration we can divide the area of a sector bounded by a curve and two radii vectors in n equal sectors. After constructing the integral curve $OP_1P_2 \cdots P_n$, we divide the vector OP_n representing the value of the whole area into n equal parts $Ob_1 = b_1b_2 = \cdots = b_{n-1}P_n$ and draw circles through the points of division $b_1, b_2, \cdots, b_{n-1}$, taking O as the center. The circles intersect the integral curve in points $p_1, p_2, \cdots, p_{n-1}$. Drawing rays from the point O through these points $p_1, p_2, \cdots, p_{n-1}$ meeting the given curve in points $a_1, a_2, \cdots, a_{n-1}$, we get the sectors $OAa_1, Oa_1a_2, \cdots, Oa_{n-1}B$ of equivalent areas. If for the construction of the integral curve we added a constant area, as it was shown in §10, we divide in n equal parts the segment $P_0'P$ (Fig. 12) representing the area of the given sector, $OP_0' = OP_0$ being the measure of the added constant area.

In a similar way we may divide the given area in n sectors proportional to arbitrary numbers dividing the vector which measures the whole area in n parts proportional to these given numbers.

12. *The influence of a change of scale.* Let OH be the length of the perpendicular drawn from the point O to the base of a triangle, OM one half the length of its base, OK and OK_1 two different units of length (Fig. 13). Constructing as in §4, we draw MN parallel to KH and MN_1 parallel to K_1H. We obtain

$$ON/OH = OM/OK, ON_1/OH = OM/OK_1 ;$$

hence $ON/ON_1 = OK_1/OK$. The lengths of the vectors which represent the value of the areas are inversely proportional to the length of the segments taken as units. As all vectors are changed in the same ratio, the integral curves give similar figures (Fig. 14).

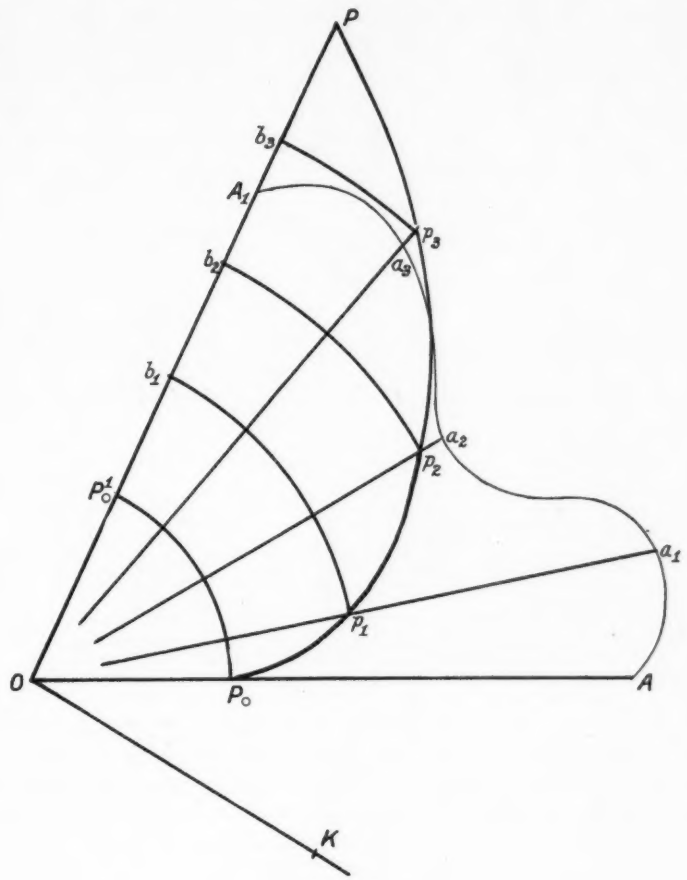

FIG. 12.

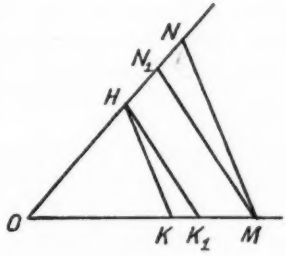

FIG. 13.

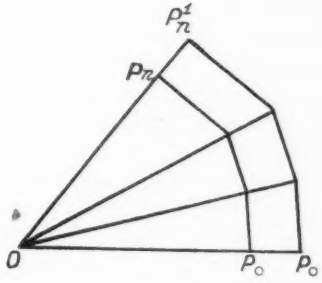

FIG. 14.

13. *Example.* As an example we shall find the positions in its orbit of a body moving according to the law of Kepler (with a constant sectorial velocity) at any given epoch (Fig. 15). Let, for instance, the orbit be an elliptic one and the

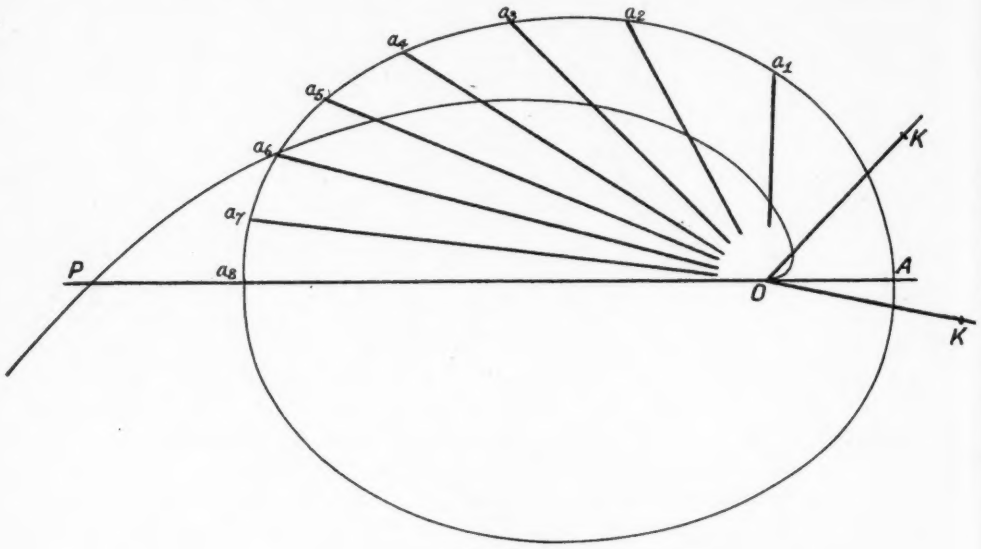

FIG. 15.

point O the focus (center of force). Let the times t and t' of any two positions of the body in its orbit be given, say at the points A and a_5. We construct the integral curve for this ellipse and we divide the radius vector of this curve measuring the sectorial area of the ellipse from the point A to the point a_5 in n equal parts, as was shown in §11. Constructing sectors with equal areas OAa_1, Oa_1a_2, Oa_2a_3, $\cdots$, Oa_4a_5, we get on the given orbit the arcs Aa_1, a_1a_2, a_2a_3, $\cdots$, a_4a_5 which the moving body passes in equal intervals of time, namely $(t'-t)/n$. We may continue constructing equivalent sectors and so extrapolate the points a_i. We may subdivide any of these intervals in as many equal subintervals as we please.

It is sufficient to construct the integral curve only for the upper half of the ellipse. To continue the integral curve we draw the rays dividing the area of the ellipse in equal parts (or any rays) and construct the vectors of corresponding length.

This method may be applied to problems of mechanics where the moving point is subject to a central force; under these circumstances it is moving according to the law of areas.

14. In some cases (in graphical integration of differential equations, for instance) it is necessary to find graphically the curve, the polar equation of

which could be written $\rho=\Phi(\theta)+C$, where $\Phi(\theta)=\int\phi(\theta)d\theta$, and $\rho=\phi(\theta)$ is the polar equation of a curve which is given graphically; C is the initial value of the integral $\int\phi(\theta)d\theta$. To make this construction we put $\phi(\theta)=\frac{1}{2}\psi^2(\theta)$ and construct the curve $\rho=\psi(\theta)$. The integral curve of this curve, $\rho=\frac{1}{2}\int\psi^2(\theta)d\theta$, is the required curve, as $\frac{1}{2}\int\psi^2(\theta)d\theta=\int\phi(\theta)d\theta$.

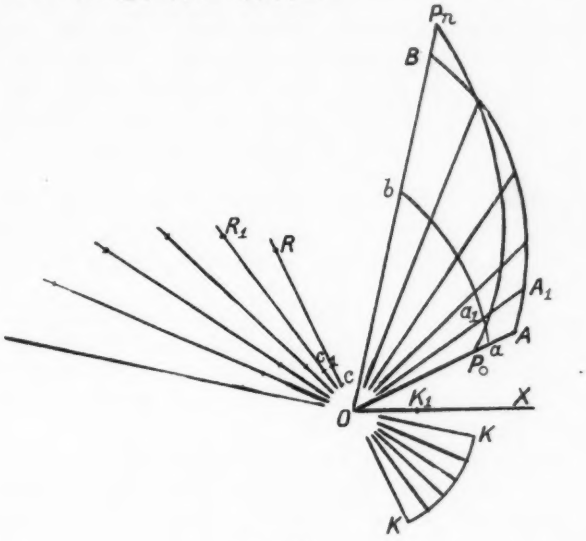

FIG. 16.

Construction of the curve $\rho=\psi(\theta)$ (Fig. 16): Let AB be the given curve, the polar equation of which is $\rho=\phi(\theta)$. Draw OR perpendicular to OA, put $OK=2$ and $OR=OA$; taking KR as diameter, draw a circle. The intersection of this circle with the ray OA, the point a, is the point of the curve $\rho=\psi(\theta)$ corresponding to the point A, as $(Oa)^2=OK\cdot OR=2\phi(\theta)$. In the same way we construct the points $a_1, a_2, \cdots, b$, corresponding to the points $A_1, A_2, \cdots, B$ and so obtain the curve $\rho=\psi(\theta)$.

Then we construct the integral curve P_0P_n of the curve ab (see §10). $OP_0=C$ is the given initial value of the integral $\int\phi(\theta)d\theta$. This curve P_0P_n represents graphically the function $\rho=\int\phi(\theta)d\theta+C$.

It is obvious that the integral $y=\int_{x_0}^{x_1}f(x)dx$ geometrically can be interpreted either as the area bounded by the curve $y=f(x)$ (in Cartesian coördinates), the X axis and the ordinates $x=x_0$, $x=x_1$, or as the sectorial area bounded by the curve $y=\sqrt{2f(x)}$ and the radii vectors $x=x_0$, $x=x_1$. (In polar coördinates, x is the polar angle and y is the radius vector). We choose the construction which suits us best. For instance, the function whose analytical expression is

$$y=\frac{1}{2}\int_0^{x_1}x^2dx=\frac{1}{6}x^3$$

may be considered as the area bounded by the parabola $y=\frac{1}{2}x^2$ (in Cartesian coördinates), the X axis and the ordinate $x=x_1$, or as the area bounded by the spiral of Archimedes $\rho=\theta$ (in polar coördinates) and the vector $\theta=\theta_1$ (the letters θ and ρ being substituted for the letters x and y). Both areas are equal.

GRAPHICAL DIFFERENTIATION

15. *Drawing of tangents.* The solution of the problem of graphical differentiation of a function represented by a curve in a polar coördinate system is based, as in the Cartesian system, on the construction of tangents to the given curve. The drawing of tangents, from the standpoint of precision, is more difficult than the substitution of the broken line for a curve required in graphical integration.

C. Runge and R. Mehmke propose several methods of drawing tangents with sufficient accuracy. Some of these methods refer to drawing tangents at a given point of the curve, others to defining the point of contact of a tangent parallel to a given direction.

C. Runge in his *Graphical methods* proposes to give the direction of a tangent and then determine the point of contact by drawing a number of chords parallel to the given direction; the intersection of the curve through the midpoints of these chords with the given curve defines the point of contact. See also methods proposed by R. Mehmke (l.c., pp. 108–111).[2]

16. *The general methods of graphical differentiation.* The problem is to find graphically the curve $\rho=\phi(\theta)$ when we are given the graph of the curve $\rho=\Phi(\theta)$, where $\Phi(\theta)=\frac{1}{2}\int_{\theta_0}^{\theta}\phi^2(\theta)d\theta$. Let OP_iP_n be the given curve (Fig. 17) and $\rho=\Phi(\theta)$

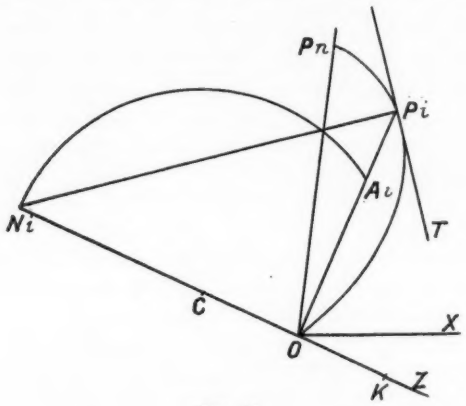

FIG. 17.

[2] There exist several instruments for drawing a tangent (or normal) at a given point of a curve, as: the "Tangentenzeichner" of A. Pflüger (see Zeitschrift des Ver. der Deutschen Ing., Bd. 58, 1914); the "Spiegellineal" of E. Reusch (see Carls Repertorium für Experimentale Physik," B. 16, 1880); the "Spiegelderivator of A. Wagener (see Physical Zeitschrift, Bd. 10, 1909). The last one is very expensive but with accuracy not less than that of other geometrical constructions.

be its equation in polar coördinates with the point O as the pole and OX the polar axis. We wish to find the point A_i of the curve $\rho=\phi(\theta)$ corresponding to the point P_i of the curve $\rho=\Phi(\theta)$, when $\Phi(\theta)=\frac{1}{2}\int_0^\theta \phi^2(\theta)d\theta$.

From the last equation we get

$$\Phi'(\theta) = \tfrac{1}{2}\phi^2(\theta), \quad \phi^2(\theta) = 2\Phi'(\theta).$$

First method: (Fig. 17). (1) Draw a tangent to the given curve at the point P_i. (2) Draw the ray OP_i and construct OZ perpendicular to OP_i. (3) On OZ lay the length OK equal to two units of length. $OK=2$. (4) Construct the normal P_iN_i to the curve at the point P_i, the point N_i being the intersection of the normal with OZ. (5) Bisect N_iK in C and taking the point C as the center describe a circle with the radius CK. This circle intersects OP_i or its prolongation at the point A_i. The point A_i is the required point of the curve $\rho=\phi(\theta)$ when $<P_iOX=\theta$, for $ON_i=\Phi'(\theta)$ (the subnormal) and $(OA_i)^2=OK\cdot ON_i=2\Phi'(\theta)$; $OA_i=\phi(\theta)$.

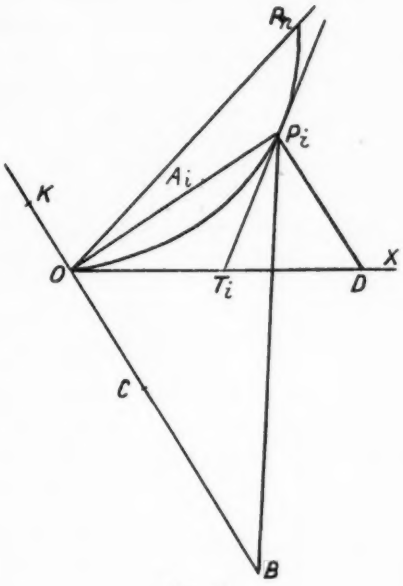

FIG. 18.

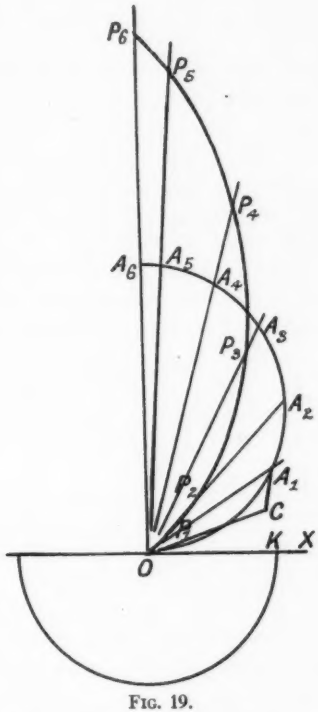

FIG. 19.

Second method: In some cases the subnormal is very long and the dimensions of the sketch do not allow the construction of the point N_i in the direction from O to N_i. We can apply then another method.

Construction (Fig. 18): P_iT_i is the tangent; $P_iD\perp OP_i$; $OB\perp OP_i$; $OK=2$; $\angle OP_iB=\angle T_iP_iD=\frac{1}{2}\pi-\mu$; $KC=CB$.

μ is the angle between the positive directions of the tangent T_iP_i and the radius vector OP_i. The point A_i is the point of intersection of the radius vector OP_i with the circle of radius CK and with the point C as center.

The point A_i is the required point of the curve $\rho=\phi(\theta)$, because $OB=OP_i\tan OP_iB=\Phi(\theta)\cot\mu=\Phi'(\theta)$ (for $\tan\mu=\rho/\rho'$).

$$(OA_i)^2=OK\cdot OB=2\Phi'(\theta);\ OA_i=\phi(\theta).$$

17. *Construction of a differential curve.* Given the curve OP_3P_6 (Fig. 19). We draw the vectors OP_1, OP_2, $\cdots$, OP_5 and construct the points A_1, A_2, $\cdots$, A_5 of the differential curve corresponding to the points P_1, P_2, $\cdots$, P_5. Through these points we draw a smooth curve which is the required differential curve.

It is impossible to draw a tangent between the points O and P_1 of the given curve. But as the vector OP_1 is approximately $\frac{1}{3}$ of the vector OP_2 we conclude that the area measured by the vector OP_1 is approximately half the area of the sector OA_1A_2. We construct a triangle OCA_1 with this area and replace the broken line OCA by a smooth curve so that the added and subtracted areas are equal.

In §18 we shall indicate more precise methods of drawing in such cases.

18. *Some secondary methods.* The inexactness of drawing tangents in some cases may be corrected by some auxiliary methods. When only the angle POX of the sector and its area represented by a segment OP are given (Fig. 20), the problem is evidently indefinite and there is an infinity of sectors satisfying these conditions. Adding some supplementary conditions, however, the problem can be made definite.

Let us consider a few such cases. We shall give the construction only, without the proof which is quite elementary. In all cases we are given the angle POX of the sector and the segment OP measuring the area of this sector.

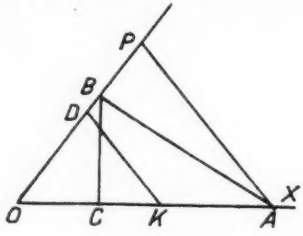

FIG. 20.

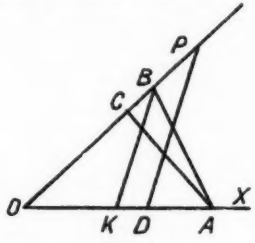

FIG. 21.

(1) It is required that the sector be represented by a triangle with the fixed end-point B (Fig. 20).

Construction: $OK=2$; $BC\perp OX$; $OD=BC$; $PA\parallel DK$. A is the vertex of the required triangle OAB.

If instead of the point B the point A is given (Fig. 21), to find the point B we construct: $OK=2$; $AC\perp OP$; $OD=AC$; $KB\parallel DP$. The point B is the required point.

(2) It is required that the sector be represented by a triangle OAB with AB parallel to a given direction $A'B'$ (Fig. 22). *Construction*: $A''P \| A'B'$; $OH' \perp A''P$; $OF = A''P$; $H'E \| FP$; $OK \perp OH'$; $OK = 2$; $OD = OE$; $KC = CD$; a circle

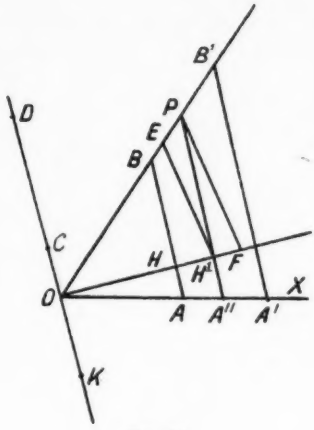

FIG. 22.

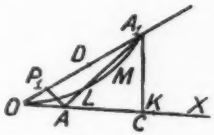

FIG. 23.

with C as the center and radius CD intersects OH' in H; a parallel to $A'B'$ through the point H gives AB. Instead of direction $A'B'$ the direction OH of the perpendicular to AB may be given.

(3) In a similar way a construction can be made when the length AB or OH is given.

These constructions can help us in drawing the differential curve or in correcting it. So in the example of §17 we can construct a triangle OA_1A (Fig. 23) the area of which is measured by the vector OP_1 and one of the vertices of which is the point A_1, found previously by the method of tangents. Then we draw the curve $OLMA_1$, so that the subtracted area OAL is equal to the added area of LMA_1.

19. *Combination of the methods.* If the given curve is of such form that only a few points, or even a single point, of contact of the curve with the tangent can be determined with sufficient accuracy, we can find the points (or the point) of the required curve $\rho = \phi(\theta)$ corresponding to these points of contact by the method of §16 and continue the construction of the curve by the methods of §18. If, for instance, the point A_i of the curve $\rho = \phi(\theta)$ (Fig. 24) corresponding to the point P_i of the given integral curve was found by means of tracing a tangent we can construct a broken line beginning with the point A_i in both directions using only the values of the radius vector of the curve OP_iP by the method of §18, case 1, without taking into consideration the tangents to the curve. To find the point A_{i-1} corresponding to the point P_{i-1} we take on the ray OP_i the value of the area of the triangle $OA_{i-1}A_i$ which is equal to the difference

of vectors $OP_i - OP_{i-1} = Op_i$ and then construct as in §18. To find the point A_{i+1} corresponding to the point P_{i+1} we take on the ray OP_{i+1} the segment $OP_{i+1} - OP_i = Op_{i+1}$ measuring the area of the triangle OA_iA_{i+1}, and so on. The broken line through these points A_i is the approximation of the differential curve. Then we draw through the point A_i a smooth curve so that the areas bounded by the broken line and this curve are equal.

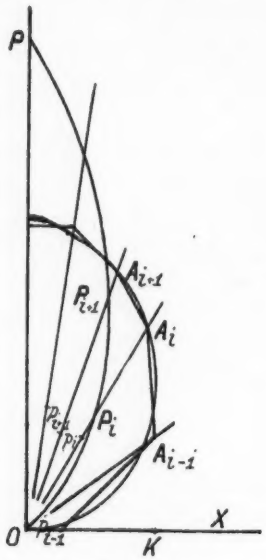

FIG. 24.

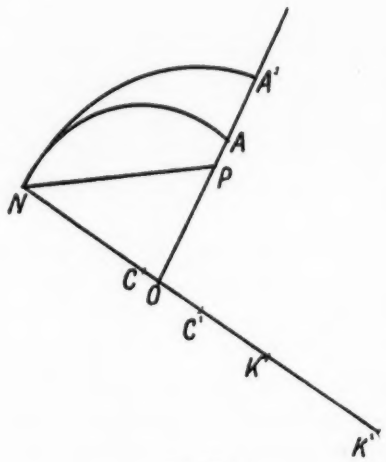

FIG. 25.

20. *The influence of a change of scale.* Let OP be the vector measuring the area of a sector (Fig. 25). By the method shown in §16 we construct the corresponding point A of the differential curve when the unit of length is represented by half the segment OK, and the point A' when the unit of length is represented by half the segment OK'. Then

$$(OA)^2 = NO \cdot OK\,; \quad (OA')^2 = NO \cdot OK'\,; \quad OA/OA' = \sqrt{(OK/OK')}.$$

The radii vectors of the differential curve are proportional to the square roots of the values of the corresponding units. We could conclude this from considering the influence of altering the scale in the case of integration (§12). Hence, if we increase the scale k times, a figure similar to the first is obtained, the linear dimensions of which are $\sqrt{k}$ times greater.

21. *Example.* Let us find the orbit of a body which is moving according to the law of constant areal velocity when one point of this orbit (Z) (Fig. 26) and the directions $(OS_1, OS_2, \cdots, OS_7)$ in which this moving body is seen in different intervals of time from the center of force (O) are known. Let the intervals of time from the first observation to the second be a_1, from the second to the third be a_2, and so on.

We take an arbitrary length OP_7 as the measure of the sectorial area bounded by the required orbit and the rays OS_1 and OS_7 and divide it in parts proportional to the numbers $a_1, a_2, \cdots$, and we draw through these points of division circles which intersect the corresponding rays $OS_1, OS_2, \cdots, OS_6$ at the points $P_1, P_2, \cdots, P_6$. In this way we obtain the integral curve $P_1P_2 \cdots P_7$ of the unknown orbit.

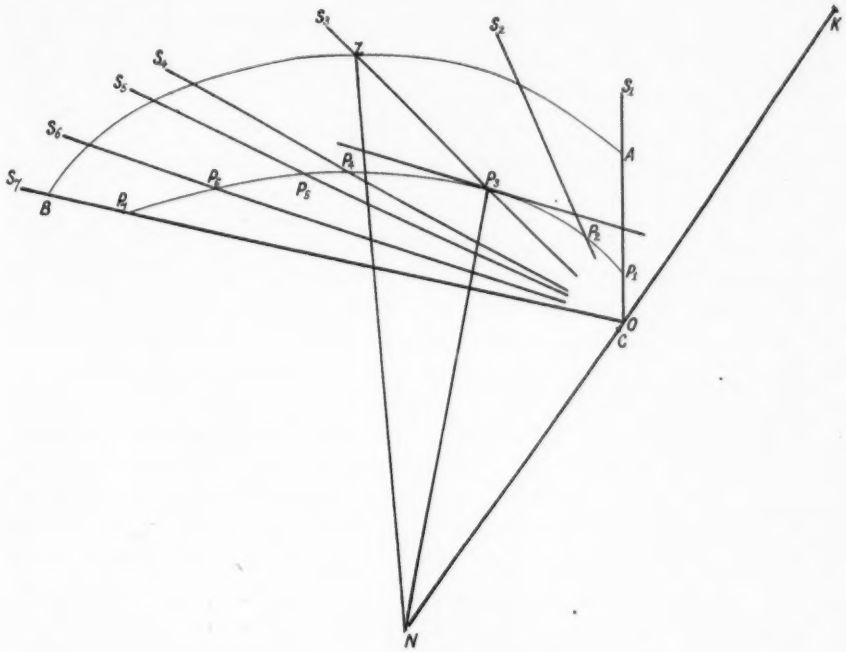

FIG .26.

The unit of length of the integral curve is not yet defined. To find this unit we draw a tangent to the integral curve at the point P_3 corresponding to the given point of the orbit, Z, the perpendicular to the radius vector OP_3, ON, the normal P_3N. Then $ON = \Phi'(\theta)$ where $\rho = \Phi(\theta)$ is the equation of the integral curve P_1P_7. We know that the point Z of the differential curve is lying on the circle through the point N and with the center on the line ON (see §16). Therefore the intersection of the mid perpendicular to the segment NZ and the line ON, the point C, is the center of this circle with the radius CN. If $CK = CN$, the point K determines the length $OK = 2$ units of length.

Then we construct the differential curve AB which is the required orbit.

If no point of the orbit is known, we shall be able to find only a figure similar to the figure bounded by the true trajectory.

HERONIAN TRIANGLES

By WM. FITCH CHENEY, JR., Tufts College

1. *Introduction.* A triangle whose sides and area are rational is called a rational triangle. For any rational triangle there exists a single infinity of other rational triangles similar to it. In any such singly infinite set of similar rational triangles there is always just one whose three sides are integers which have no factor common to all three. This we shall call the Heronian triangle of the set. It is so named after Hero, who is credited with the formula for the area, K, of a triangle of sides a_1, a_2, a_3:

$$K = (ss_1s_2s_3)^{1/2}, \quad \text{where} \quad s = \tfrac{1}{2}\sum_{i=1}^{3} a_i \quad \text{and} \quad s_i = s - a_i.$$

While some authors admit in the class of plane triangles those which have zero area and a zero altitude, such flat triangles are not included in the theorems of this article. The class of Heronian triangles is then to include all triangles of non-zero, rational area and integral sides which have no factor common to all three. This paper presents some theorems regarding factors of the sides and area of Heronian triangles.

2. *Rational triangles.* In the July, 1921, issue of this Monthly, Professor L. E. Dickson of the University of Chicago showed that all shapes of rational triangles could be formed by the juxtaposition of pairs of rational right triangles.[1] This process leads, however, to duplication, producing the same shape of rational triangle in three different ways, depending on which altitude of the oblique triangle is the line of juxtaposition between the right triangles.

Since for any triangle of sides a_i and angles A_i, $\tan \frac{1}{2}A_i = K/ss_i$, the tangents of the half-angles of any rational triangle must all be rational. If we set $t_i = \tan \frac{1}{2}A_i$, we know from trigonometry that $t_1t_2 + t_2t_3 + t_3t_1 = 1$. Hence if we have a triangle for which we know two t's rational, the third must also be rational, and the triangle will be similar to a Heronian triangle.

A triangle T can be constructed similar to any given triangle and having the vertex of its largest angle in the first quadrant of a set of rectangular coordinates, and its other two vertices at $(-1, 0)$ and $(1, 0)$. For any given triangle this triangle, T, is unique. The vertex of its largest angle must be within or on the circle centered at $(-1, 0)$ with radius 2, and consequently its incenter must be within or on the loop of the cubic curve

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[1] So far as the author has been able to determine, the only information hitherto published about Heronian triangles is typified by the article of Professor Dickson, referred to above, and by chapter fifteen of the first volume of Dr. Hermann Schubert's *Mussestunden*. The standard method has been to construct a Heronian triangle by the juxtaposition of two right triangles, each having three integral sides. But there are Heronian triangles which can not be so constructed, as for example, one of sides 25, 34 and 39 units. The author has found no reference to the nature of the factors of the sides of Heronian triangles, which consideration constitutes the major portion of the present paper.

$$y^2 = (1 + x)(1 - x)^2(3 - x)^{-1},$$

and also in the first quadrant. This cubic curve is the locus of the intersections of the angle-bisectors of all triangles having two vertices at $(-1, 0)$ and $(1, 0)$ and the third on the circle $(x+1)^2+y^2=4$.

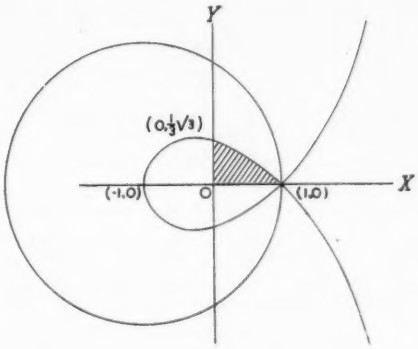

FIG. 1.

Consequently we may take any number x such that $0 \leqq x < 1$, and with it any positive number $y \leqq (1-x)\ (1+x)^{1/2}\ (3-x)^{-1/2}$, as the coordinates of the incenter of a triangle.

Theorem 1. *Each different point P of the shaded area in the accompanying diagram determines uniquely a differently shaped triangle T, of which it is the incenter.*

Since the bisectors of the interior angles of T all pass through P, the tangents t_i of the half-angles are expressible rationally in terms of the coordinates of P, and conversely.

Theorem 2. *The necessary and sufficient condition that T be a rational triangle is that the coordinates of P shall be rational.*

Thus all different shapes of rational triangles are obtained once each by letting P stop once on each point of the shaded area in the diagram which has rational coordinates. Since the set of all points with rational coordinates is denumerable, we have the two theorems:

Theorem 3. *The set of all Heronian triangles is denumerable.*

Theorem 4. *The set of all rational triangles is denumerable.*

3. *Properties of the t_i.* We have shown that any point P of a denumerable set of points furnishes the three t_i of a uniquely determined set of similar rational triangles. This set contains just one Heronian triangle. To this latter triangle we shall now confine our attention. If its sides be a_i $(i=1, 2, 3)$, its area K, its semi-perimeter s, and $s_i=s-a_i$, Hero's formula then reads:

$$(A) \qquad K = (ss_1s_2s_3)^{1/2}.$$

We also know from trigonometry that

$$(B) \qquad K = ss_it_i$$

and that

$$(C) \qquad t_1t_2 + t_2t_3 + t_3t_1 = 1.$$

Obviously

$$(D) \qquad s = \sum_{i=1}^{3} s_i.$$

Dividing the cube of (B) by the square of (A) gives

$$(E) \qquad K = s^2t_1t_2t_3,$$

which we may combine with (B) to eliminate K and get

$$(F) \qquad s_i = st_jt_k \quad (i,j,k = 1,2,3\,;\ i \neq j \neq k).$$

Replacing s_i in (F) by its definition value $s-a_i$, we may obtain

$$(G) \qquad a_i = s(1 - t_jt_k).$$

Since the sides of our triangle are each positive, this gives:

Theorem 5. *The product of any two t's must be less than* 1.

4. *The numerators and denominators of rational* t_i. Now since each t_i in any Heronian triangle is rational, we shall write it as a fraction *in lowest terms*,

$$(H) \qquad t_i \equiv n_i/d_i \qquad (i = 1,2,3),$$

where d_i and n_i have no common factor. Setting (H) into (C) gives

$$(I) \qquad d_1n_2n_3 + d_2n_3n_1 + d_3n_1n_2 = d_1d_2d_3.$$

By transposition and division (I) may be converted into

$$(d_jd_k - n_jn_k)/n_i = (d_jn_k + d_kn_j)/d_i,$$

and since the denominators of these two equal fractions have no common factor, each fraction must reduce to an integer. This integer we shall call v_i. Accordingly we write

$$(J)\ (1) \qquad d_iv_i = d_jn_k + d_kn_j \qquad \text{and } (2) \qquad n_iv_i = d_jd_k - n_jn_k.$$

If these two equations be squared and added, there results

$$(K) \qquad v_i^2(d_i^2 + n_i^2) = (d_j^2 + n_j^2)(d_k^2 + n_k^2).$$

Solving for v_i and deriving v_j by symmetric permutation of subscripts produces the remarkably fruitful equation

$$(L) \qquad v_iv_j = d_k^2 + n_k^2.$$

Now it is well known in number theory that the sum of two relatively prime squares can only have factors each of which is the sum of two relatively prime squares. For our purposes this means:

Theorem 6. *Every factor of each v is expressible as the sum of two relatively prime squares.* In other words, no v has any factor of the form $4n-1$. Further, from (L) follows:

Theorem 7. *Each v is prime to each of the d's and n's whose subscripts differ from its own.* A logical consequence of this is:

Theorem 8. *Any factor common to two v's is prime to all the d's and n's.*

Now since every d and n is 1 or greater, the minimum value of d^2+n^2 is 2. (In this case only, our Heronian triangle is a right triangle.) In this light, (L) demonstrates:

Theorem 9. *In any Heronian triangle at least two v's exceed* 1.

If the three v's have a common factor f, it must by $(J, 1)$ divide $d_in_k+d_kn_j$, and hence must also divide $\sum n_i(d_in_k+d_kn_j)$. But by (I) this equals $2d_1d_2d_3$. By theorem 8, f is prime to all the d's, and must therefore be either 1 or 2. If f were 2, then by theorem 8, all the d's and n's would have to be odd. This would make the left member of (L) a multiple of 4 while the right side would be twice an odd number, which is impossible. Hence the supposition that f might be 2 can not hold, and there results:

Theorem 10. *The three v's can not have any common factor.*

If the t's in (G) be replaced by their values from (H) and the result reduced by $(J, 2)$, it appears that

$$(M) \qquad a_i = sn_iv_i/d_jd_k,$$

which must be an integer. A similar substitution of (H) in (E) gives

$$(N) \qquad K = s^2n_1n_2n_3/d_1d_2d_3,$$

which we shall use later.

5. *The factors of d_i and n_i.* Since any factor common to three terms must divide their sum, and since each d is prime to its corresponding n, there follow from (I) three important theorems:

Theorem 11. *Any factor common to two n's divides the third d.* We shall represent the greatest common factor of d_i, n_j, and n_k by e_i.

Theorem 12. *Any factor common to a d and an n divides the third n.*

Theorem 13. *Any factor common to two d's must divide the third d.* We shall represent the greatest common factor of the three d's by E. In the most general case we may then write:

$$(O) \quad (1) \qquad d_i \equiv c_ie_iE \qquad \text{and} \quad (2) \qquad n_i \equiv b_ie_je_k,$$

where i, j, k represent the three different numbers 1, 2, 3 in some order. More specifically,

$$(O') \qquad t_1 \equiv b_1e_2e_3/c_1e_1E, \qquad t_2 \equiv b_2e_3e_1/c_2e_2E, \qquad t_3 \equiv b_3e_1e_2/c_3e_3E,$$

where each fraction is in lowest terms, and no two of the b's and c's have any common factor.

If we substitute from (O) into (M) and cancel, we get

$$(P) \qquad a_i = sb_iv_i/c_jc_kE^2.$$

By the definition of a Heronian triangle, the three a's are integers with no factor common to all three. And since b_iv_i is prime to $c_jc_kE^2$, it follows that $c_1c_2c_3E^2$ must equal the numerator of s. Now the denominator of s must be either 1 or 2, and the latter case may be shown impossible; for if the denominator of s were 2, then b_1v_1, b_2v_2, and b_3v_3 would all have to be even to satisfy (P). But by definition, only one b can be even, and since by theorem 7 the other two v's must then be odd, we are led to an absurdity. The only alternative is

$$(Q) \qquad s = c_1c_2c_3E^2,$$

which proves:

Theorem 14. *The semiperimeter of any Heronian triangle is an integer.* Since the sum of the three sides is thus even and the three have no common factor, there also appears:

Theorem 15. *Any Heronian triangle has just one even side.*

Substituting (O) and (Q) in (M) and (N) gives, by cancelling,

$$(R) \qquad a_i = b_ic_iv_i,$$

$$(S) \qquad K = b_1b_2b_3c_1c_2c_3e_1e_2e_3E,$$

which last leads at once to the theorems:

Theorem 16. *The area of any Heronian triangle is an integer.*

Theorem 17. *The area of any Heronian triangle is the least common multiple of the d's and the n's.*

We shall now show that the factor 2 must appear in the area. If the area were odd, all the d's and n's would have to be odd. Now $(d_1+n_1)(d_2+n_2)$ is the product of two even numbers, and hence a multiple of 4, while n_1n_2 is odd, so that $(d_1+n_1)(d_2+n_2)-2n_1n_2$ must be twice an odd number. But this equals the sum of the two even numbers $d_1d_2-n_1n_2$ and $d_1n_2+d_2n_1$, so that one of them is twice an odd number while the other is a multiple of 4. Then by (J), v_3 is twice an odd number and either d_3 or n_3 must be even, contrary to the assumption that all the d's and n's were odd. We may consequently state:

Theorem 18. *The area of any Heronian triangle is even.*

We next show that the factor 3 must appear in the area. We first assume the contrary and then show that such an assumption leads to a contradiction. If 3 were not a factor of the area, no d or n could be a multiple of 3 (by theorem 17). Now the squares of all numbers prime to 3 are congruent modulo 3, so that $d_1^2-n_1^2$ and $d_2^2-n_2^2$ would be multiples of 3. Then $(d_1^2-n_1^2)d_2n_2+(d_2^2-n_2^2)d_1n_1$ would be a multiple of 3. But this equals $(d_1d_2-n_1n_2)(d_1n_2+d_2n_1)$, which by (J) equals $d_3n_3v_3^2$. Since by theorem 6, v_3 can not be a multiple of 3, either d_3 or n_3 would then have to be a multiple of 3, contrary to the assumption that no d or n could be a multiple of 3. As a result of this consideration we state:

Theorem 19. *The area of any Heronian triangle is a multiple of 3.*

We shall next conduct a similar investigation of the factor 5. If the area is not a multiple of 5, then by theorem 17, all the d's and n's are prime to 5. Now the fourth powers of all numbers prime to 5 are congruent modulo 5. Consequently 5 divides $d_1^4 - n_1^4$, which by (L) equals $(d_1^2 - n_1^2)v_2v_3$. Then $(d_1^2 - n_1^2)d_2n_2v_1v_2v_3$ is a multiple of 5, and so similarly is $(d_2^2 - n_2^2)d_1n_1v_1v_2v_3$. Then their sum,

$$[(d_1^2 - n_1^2)d_2n_2 + (d_2^2 - n_2^2)d_1n_1]v_1v_2v_3,$$

which equals

$$(d_1d_2 - n_1n_2)(d_1n_2 + d_2n_1)v_1v_2v_3$$

and which reduces by (J) to $d_3n_3v_1v_2v_3^3$ must also be a multiple of 5. Hence if the factor 5 does not appear in any of the d's or n's, it must appear in one of the v's. This gives:

Theorem 20. *In any Heronian triangle, if the area is not a multiple of* 5, *one of the sides is.*

From (R) we see that any factor common to a_i and a_j must divide v_i and v_j, so that by theorem 6 we may state:

Theorem 21. *Every factor common to two sides of a Heronian triangle must be expressible as the sum of two relatively prime squares.* Furthermore, by theorem 8, we state:

Theorem 22. *No factor common to two sides of a Heronian triangle can be a factor of the area.* Also, by (R) and (S), we have:

Theorem 23. *Any number of the form* $4n-1$ *which is a factor of a side of a Heronian triangle, must also be a factor of the area.*

6. *Construction of Heronian triangles.* Probably the easiest way to make a list of Heronian triangles is to develop the method suggested in paragraph 2. We may choose any two fractions in lowest terms for t_1 and t_2, subject to the conditions that $0<t_1\leqq t_2<1$ and $t_1\leqq(1-t_2^2)/2t_2$. This keeps the point P of paragraph 2 in the shaded area of the loop in the diagram. Since Heronian triangles form a denumerable set, we may put them in the following order:

(a) We define the index, m, of a Heronian triangle as $d_1+n_1+d_2+n_2$.

(b) Of two triangles of different indices, that with the smaller index shall precede.

(c) Of two triangles of the same index but different t_1's, that with the smaller t_1 shall precede.

(d) Of two triangles of the same index and the same t_1, that with the smaller t_2 shall precede.

Having given t_1 and t_2, we first compute t_3 from (C). We next use (J) to obtain the v's, determine the e's and E from their definitions under theorems 11 and 13, and thence derive the b's and c's. We are then in a position to write the a's and K from (R) and (S).

The following is a list of all Heronian triangles for which the index m does not exceed 12:

m	t_1	t_2	t_3	v_1	v_2	v_3	E	e_1	e_2	e_3	b_1	b_2	b_3	c_1	c_2	c_3	a_1	a_2	a_3	K
6	1/2	1/2	3/4	5	5	1	2	1	1	1	1	1	3	1	1	2	5	5	6	12
7	1/3	1/2	1/1	1	2	5	1	1	1	1	1	1	1	3	2	1	3	4	5	6
8	1/4	1/2	7/6	5	17	1	2	1	1	1	1	1	7	2	1	3	10	17	21	84
	1/3	1/3	4/3	5	5	2	3	1	1	1	1	1	4	1	1	1	5	5	8	12
9	1/5	1/2	9/7	5	26	1	1	1	1	1	1	1	9	5	2	7	25	52	63	630
	1/4	1/3	11/7	10	17	1	1	1	1	1	1	1	11	4	3	7	40	51	77	928
	1/3	2/3	7/9	13	10	1	3	1	1	1	1	2	7	1	1	3	13	20	21	126
10	1/6	1/2	11/8	5	37	1	2	1	1	1	1	1	11	3	1	4	15	37	44	264
	1/5	1/3	7/4	5	13	2	1	1	1	1	1	1	7	5	3	4	25	39	56	420
	1/4	1/4	15/8	17	17	1	4	1	1	1	1	1	15	1	1	2	17	17	30	120
	1/4	2/3	10/11	13	17	1	1	2	1	1	1	1	5	2	3	11	26	51	55	660
	2/5	1/2	8/9	5	29	1	1	1	2	1	1	1	4	5	1	9	25	29	36	360
11	1/7	1/2	13/9	5	50	1	1	1	1	1	1	1	13	7	2	9	35	100	117	1,638
	1/6	1/3	17/9	10	37	1	3	1	1	1	1	1	17	2	1	3	20	37	51	306
	1/5	1/4	19/9	17	26	1	1	1	1	1	1	1	19	5	4	9	85	104	171	3,420
	1/5	2/3	1/1	1	2	13	1	1	1	1	1	2	1	5	3	1	5	12	13	30
	1/3	2/5	13/11	29	10	1	1	1	1	1	1	2	13	3	5	11	87	100	143	4,290
	1/2	3/5	7/11	34	5	1	1	1	1	1	1	3	7	2	5	11	68	75	77	2,310
12	1/8	1/2	3/2	1	13	5	2	1	1	1	1	1	3	4	1	1	4	13	15	24
	1/7	1/3	2/1	1	5	10	1	1	1	1	1	1	2	7	3	1	7	15	20	42
	1/6	1/4	23/10	17	37	1	2	1	1	1	1	1	23	3	2	5	51	74	115	1,380
	1/6	2/3	16/15	13	37	1	3	2	1	1	1	1	8	1	1	5	13	37	40	240
	1/5	1/5	12/5	13	13	2	5	1	1	1	1	1	12	1	1	1	13	13	24	60
	1/4	2/5	18/13	29	17	1	1	2	1	1	1	1	9	2	5	13	58	85	117	2,340
	1/4	3/4	13/16	25	17	1	4	1	1	1	1	3	13	1	1	4	25	51	52	784
	2/7	1/2	12/11	5	53	1	1	1	2	1	1	1	6	7	1	11	35	53	66	924
	1/3	3/5	6/7	17	5	2	1	3	1	1	1	1	2	1	5	7	17	25	28	210
	2/5	2/3	11/16	13	29	1	1	1	1	2	1	1	11	5	3	8	65	87	88	2,640

It is interesting to note, in the foregoing list of the first twenty-eight Heronian triangles, that only two are right triangles. (A right Heronian Triangle is characterized by $t_3 = 1$.)

Although each of the triangles listed above has at least one altitude which is an integer, this is not always true, as may be seen in the case of the Heronian triangle whose sides are 39, 58, and 95 and whose area is 456.

HOW CAN INTEREST IN CALCULUS BE INCREASED?[1]

By ROSCOE WOODS, University of Iowa

As teachers I presume that all of us have certain aims in our work. There is no doubt but that these aims vary with the individual. However, one of the important facts to keep before ourselves is that mathematics, due to its inherent nature, needs a teacher more than any other subject. If we do not have definite aims of a high type as to our functions as teachers, we are not going to create much enthusiasm for our subject. Unless we are convinced that the

[1] Read before the Iowa Section of the Mathematical Association of America, May 4, 1928.

study of mathematics can do something for the student beyond the acquisition of certain mechanical skills and facts required for the next course and unless we believe that the student acquires certain fundamental thought processes from mathematics, we will not engender a great deal of interest in our students.

The study of mathematics ought to help the student to express himself in clear, forcible, and concise language. It should unfold his rational faculties and teach him how to use them. In the various subjects that we set before him and in the problems that he solves, he ought to be able to penetrate from the surface to the central idea. We as teachers should be convinced and should teach that one of the primary aims of all learning, whether of a mathematical nature or not, is to see relations that exist between things and to correlate facts with principles that seem to be unrelated. This is another way of saying that mathematics should be so taught that the student is brought to realize that he has faculties that can discern the good and the bad intellectually and that this discernment is the foundation of the development of character for which all education should exist.

In the teaching of the calculus these general aims should be more nearly attained than in any other single subject. Perhaps, to be more specific, we ought to ask ourselves why we teach calculus, to liberal arts students, in preference to some other subject which would contribute toward their culture and mental development. The answer to this question presents a multiplicity of notions. In a broad way we are accustomed to think of the practical side of calculus, when engineers are mentioned; of its applications, when other science students are named; and of the cultural side, when liberal arts students are considered. Let us enumerate some of the aims for teaching calculus. Do we teach it with the idea that the student will recognize forms and acquire facts and technique? Do we teach it with the idea that it unifies all that the student has learned before and that it is the key stronghold from which all other fields of mathematics are to be explored? Do we teach it solely as a preparation of the engineer and of the teacher of elementary mathematics? Do we teach it as a means of interpreting physical phenomena? Do we teach it simply as a tool whereby other achievements are accomplished? Or do we teach it as a formal exercise in logic with the idea that it may stimulate certain thought processes?

Some years ago Professor Osgood of Harvard,[2] in an address before the American Mathematical Society, maintained that the calculus should be taught as a means of interpreting physical phenomena. He claimed that there was only one calculus for all students, no matter what the ultimate aim or profession of the student. He presents the course formally and uses the differential as soon as it is possible to introduce it. Professor Rietz has likewise written an article, which appeared in this Monthly,[3] in which he points out that the aim in teaching calculus should be to implant certain notions that the student should

[2] W. F. Osgood, Bulletin of the American Mathematical Society, vol. 13, p. 449.

[3] H. L. Rietz, this Monthly, vol. 26 (1919), p. 341.

remember ten years afterward. Professor Bliss of Chicago[4] has written a monograph on the subject, *The function concept and the calculus*. All of these papers are well worth reading and will help you to decide why the calculus should be taught.

At first sight it seems that it ought to be a fairly simple matter for us to inculcate in the student the three central ideas of the calculus; namely, the derivative, the anti-derivate or the indefinite integral, and the definite integral interpreted as the limit of a sum. But the appreciation of these ideas presupposes a considerable amount of preparation on the part of the student. For the study of the calculus, work, intelligence and a certain spirit of adventure are needed. To pursue this study properly and with the greatest pleasure, the student must also possess the spirit of exploration. A few students possess this spirit but the general attitude is a passive one.

Some of the factors that give rise to the lethargic attitude are: (1) the trend of high school instruction, (2) preparation of the teachers of elementary mathematics, (3) training of college freshman, (4) the fact that no organized attempt is made to acquaint students with the content and purposes of the calculus, (5) the type of text book at our disposal, and (6) the general attitude and habits of the American student. Let us examine these six reasons briefly.

(1) The best index to the trend in high school instruction is to note the quality of the product produced. The way in which the high school graduate handles his complex simplification problems, the manner in which he analyzes problems in story form, and his feeble attempt to give definitions and logical explanations accurately usually indicate a lack of training rather than a lack of intelligence. Also there are certain habits that are indicative of training. For instance, the failure to read carefully the text is very common and usually leads to poor grades in college. This probably arises from the fact that the problems assigned in high school were of a single type and needed only the application of the illustrative exercise. Let us give another illustration of habit formation. Very few high school students have any idea how to study for recitation or how to prepare for a written lesson or how to use their time during the examination period. In other words, the fad of omitting the hard topics or of giving them little emphasis seriously handicaps the student's future.

(2) In general our elementary teachers are not highly trained. For instance only 47 percent in this state have had training in calculus. We should blush with shame when we compare the training of our elementary teachers with that of teachers in Europe.[5] There is no surer way of dampening a student's ardor for mathematics than to give him a teacher who is unprepared and who has no inherent interest in mathematics. We know that it is not uncommon for a

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[4] G. A. Bliss, in *Monographs on Modern Mathematics* (1915), Edited by J. W. A. Young.

[5] Report of the National Committee on *The Reorganization of Mathematics in Secondary Education*, Chapter 14.

high school principal to ask a teacher who majored in English, for instance, to take a mathematics class in the high school. There seems to be an impression abroad that anyone can teach mathematics and just as long as this impression prevails, just so long will this vicious practice of assigning majors in other fields to teach mathematics classes continue.

(3) There are very few freshmen who are adequately prepared for and who can successfully master college algebra, trigonometry, and analytical geometry in one year, four periods per week. It requires time to acquire mathematical ideas and much training for mastery. To be frank we shall have to admit that college algebra is a conglomeration of topics. We admit that they are needed and badly needed; but they are not built around a central idea. I refer to the rational functions. In trigonometry we give the student a smattering of the circular functions, but we do not let him hear of their inverses. Analytical geometry offers the first opportunity to the student to use all his mathematical knowledge. It is the proving ground. The average student has just about enough time to learn the terminology connected with the course, but not enough of the technique required for enjoyment. He learns that there are such things as the straight line and the standard forms of the conics. He is allowed a glimpse of polar coordinates, higher plane curves, and transformation of coordinates. His solid analytical geometry is sealed with a great seal. He finishes his course with no idea of what the fundamental problem of analytical geometry is. He has graphed a few curves but he has not been given time to study the graphs after they have been made.

(4) We never point out to the student that all his work thus far has to do with functions of different kinds. We never classify them for him. He has been learning some of their properties through algebraical and graphical methods, but he has not been brought face to face with the tangent problem in his analytics, nor have his algebra and trigonometry convinced him that he needs to have a method for studying the variation of these functions. At no stage in the freshman's preparation do we acquaint him with the aims and purposes and the grandeur of the calculus. If he elects mathematics in the second year, he faces the same kind of useless (?) computation that he has done all his life with no apparent reason or incentive. Mathematical enthusiasm does not thrive in such soil.

(5) The absence of good exposition in our calculus texts is apparent. In order to meet the demand of the technical schools and of economy our texts are a sort of a hybrid between a good text and a mere handbook. If the teacher is not careful his students finish calculus not knowing the classes of the functions that have been studied. Unless the teacher is diligent, the student has not even found out the fundamental problems of the course, and his attitude toward advanced calculus is similar to that which he had for his calculus when he finished analytics.

(6) Finally, the student's attitude toward learning for learning's sake needs a revision. What student thinks of studying when not in school? Money can

be made with little study, and a position can be secured through channels other than that of scholastic attainment. The question uppermost in every student's mind is, "Of what practical value is this?" The information he seeks is that which he thinks will enable him to make ready money. But so long as the colleges and universities grant diplomas by the wholesale, why should we blame the student if he desires a diploma rather than the possession of that for which the diploma stands.

How can interest in calculus be increased? Clearly by trying to remedy the conditions set forth above. The following suggestions are offered. (1) Take every opportunity available to convince our fellow teachers outside of mathematics that we have a live, growing science. (2) Make more use in our elementary classes of the history of mathematics and the biographies of mathematicians both dead and living. (3) Be sure that our mathematics majors and advanced students know the various divisions of our science and its importance in the world. (4) Study carefully the problem of the dependence of all the sciences upon mathematics and make these results available. (5) Rearrange our courses to meet changed conditions, giving more credit where credit is due. (6) Prepare a series of charts, on various topics, that can be used in the classroom, especially one showing divisions of our science. (7) Assemble all our students by classes for general lectures along broad and unifying lines. (8) Consider some means on the part of the Iowa Section of the Association of recognizing the ability of any outstanding student who has finished the calculus. (9) Finally, let the Mathematical Association of America, the mathematics sections of the State teachers' associations, and the National Council of Teachers of Mathematics bend every effort to keep mathematical instruction in the hands of those prepared to give it.

GENERALIZATIONS OF THE THEOREM OF PYTHAGORAS AND EUCLID'S THEOREM OF THE GNOMON[1]

WILLIAM J. HAZARD, University of Colorado

Theorem 1, *Fig.* 1:

If any parallelogram P be drawn with its vertices $abcd$ in the sides of another parallelogram Q which has vertices $efgh$, the area of P equals the sum of parallelograms $afij$ and $djkh$ which are formed by drawing di and ak parallel to the respective sides of Q.

Outline Proof:

Move cb along itself to position mn. Move dm along itself to position jo. Then parallelogram $abcd = anmd = anoj = afij + fnoi$. Triangle $cbg = mno$ and mbi is common to them. Hence trapezoid $bnoi = migc$. But triangle $fnb = pmc$, hence parallelogram $fnoi = pigc = djkh$. Hence $P = afij + djkh$.

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[1] Presented to the Rocky Mountain Section of the Mathematical Association of America, April, 1927.

This theorem generalizes the theorem of Pythagoras in three ways, for if we first assume Q to be a rectangle, then a square, and finally make $cg=dh$, we shall have Fig. 2 which represents the Hindu proof[2] of the theorem of Pythagoras as an extremely special case of our theorem.

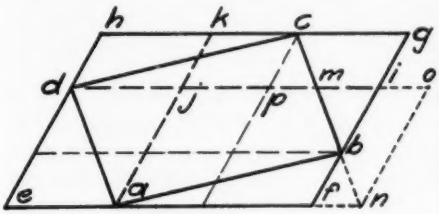

Fig. 1

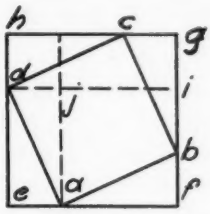

Fig. 2

Theorem 2, *Fig.* 3:

If parallelogram P be drawn inside parallelogram Q with a common diagonal ac, the area of P will equal the difference of parallelograms $ofid$ and $jdlh$ which are formed by drawing ji and ol through the vertex d of P, parallel to the respective sides of Q.

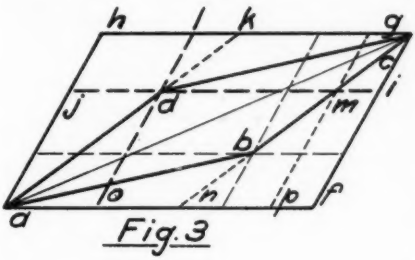

Fig. 3

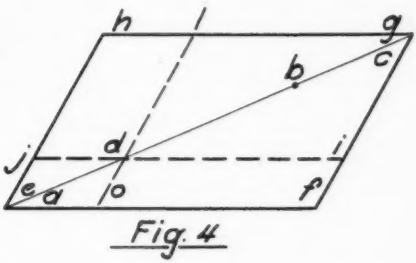

Fig. 4

Outline Proof:

Move cb along itself to position mn. Move an along itself to position op. Then parallelogram $abcd=anmd=opmd=ofid-pfim$. But triangle $nfg=kha$; $npm=dja$ and $mig=kld$, hence parallelogram $pfim=jdlh$ and $P=ofij-jdlh$.

Assume the vertices b and d in Fig. 3 to be moved parallel to the ends of Q (or otherwise) until they fall in the diagonal ge as in Fig. 4. The area of P has been thus reduced to zero and $ofid=jdlh$. Then our theorem No. 2 is seen to be a generalization of Euclid's theorem of the gnomon, Book I, Prop. 43, "In any parallelogram the complements of the parallelograms about the diagonal are equal to one another."

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[2] See Rao, *Paper Folding*, or Heath's *Euclid*, I, 47, notes.

In Fig. 4, *eodj* and *digl* are the "parallelograms about the diagonal," and the other two are the "complements."[3]

The transition from Fig. 1 to Fig. 3 may be observed by noting that in Fig. 1, when c approaches g, a approaches e. At the limiting position $djkh$ is reduced to zero and P equals $efid$ which now coincides with $afij$. When vertices b and d are now moved in parallel to ef, the area of P decreases more rapidly than the area of $ofid$ in Fig. 3, and thus necessitates the subtraction of the parallelogram $jdlh$ as shown.

The theorem of the gnomon may be extended to space by considering any parallelopiped as in Fig. 5, with three planes passed through a point p of the diagonal bh, parallel to the respective faces. The diagonal bh is projected as the diagonal bd in the parallelogram $abcd$, and it is projected as the diagonal be in the parallelogram $abfe$. The shorter segment bp of the diagonal bh is the diagonal of the parallelopiped ilp. The longer segment ph is the diagonal of the parallelopiped psh. Following Euclid's language, these two are the "parallelopipeds about the diagonal." The "complements" of ilp are ikm, jln, and prf. These three complements are equal.

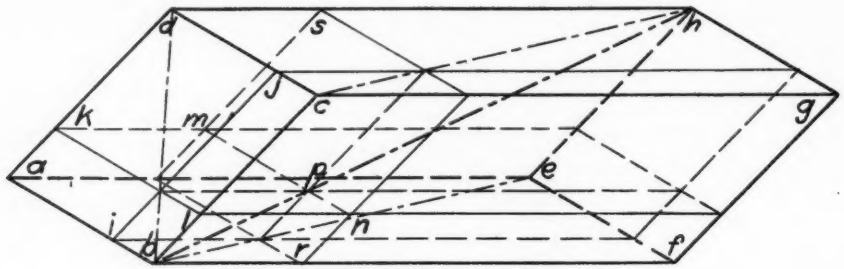

Fig. 5

Theorem 3, Fig. 5:

Outline Proof:

By the plane theorem of the gnomon, face ki = face jl. Hence parallelopiped $ikm = jln$ for $km = ln$. Similarly, face cn equals face nf and $jln = prf$ as they have equal thickness, pn. It will be observed that these three "complements" have each one face in common with the parallelopiped ilp. Similarly, the three parallelopipeds having each one face in common with psh are equal.

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[3] See Geo. W. Evans, *Some of Euclid's Algebra*, The Mathematics Teacher, Mar., 1927.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

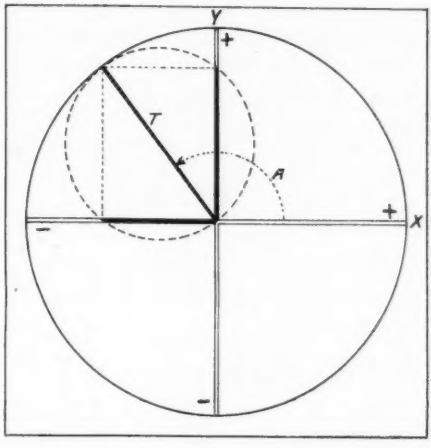
The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. A MODEL FOR SHOWING SINES AND COSINES

WILLIAM J. HAZARD, University of Colorado

When discussing the line values of the sine and cosine of an angle, it is common practice to assume a unit length for the terminal side of the angle and to show that the sine is then the ordinate of the end of the terminal line and the cosine is the projection of the abscissa of the same point on the X axis or the projection of the terminal side upon the X axis. We may just as well consider the sine also as the projection of the terminal side of the angle upon the Y axis. This assumption is made in constructing the model shown in the figure.



The X and Y axes are slots in the face of the model. The circle with unit radius T is drawn in ink on the face of the model. The terminal side T of the angle A is a stiff pointer which is secured to a knob at the back of the model and can be rotated continuously about the origin. A white disc the size of the large circle is placed back of the faceplate and secured rigidly to the knob which rotates the pointer. On this white disc is a completely blackened circle (shown dashed) which has T for a diameter. Thus the pointer T in front of the faceplate and the black circle behind the faceplate rotate about the origin, maintaining their positions in relation to each other.

Presented to the Rocky Mountain Section of the Mathematical Association of America, April, 1928.

That part of the black disc showing through the slots is equivalent to the projections of T upon the axes and hence for every possible value of the angle A, shows the sine and cosine with the proper algebraic sign. A slow and uniform rotation of T around the circle shows the relative rate of variation of the functions as well as their values. The axes are divided decimally and readings may be taken to show numerically that the sum of the squares of sine and cosine is unity. When discussing the functions of any angle A plus or minus ninety degrees, it is only necessary to give the whole model a quarter turn to show the actual interchange of ordinate and abscissa. This makes very evident, the reason for using the cofunctions of A.

II. AN EXTENSION OF THE BERNOULLI SUMMATION FORMULA

By H. S. VANDIVER, University of Texas

The Bernoulli summation formula,[1]

$$1^n + 2^n + \cdots + (k-1)^n = [(b+k)^{n+1} - b^{n+1}]/n + 1,$$

where in the expansion of $(b+k)^{n+1}$ we replace b^i by b_i, and where also $b_0 = 1$, $b_1 = -1/2$, $b_{2i} = (-1)^{i-1}B_i$, $b_{2i+1} = 0$, $i > 0$, the numbers $B_i = 1/6$, $B_2 = 1/30$, etc. being the numbers of Bernoulli, is well known. It is not so well known, however, that this summation formula admits of a simple extension to integers in arithmetical progression which is apparently due to Glaisher.[2]

I shall give here a proof which is a bit different from Glaisher's. Consider the known relation, using the symbolic method of Blissard and Lucas,[3]

$$(x+1+b)^{n+1} - (x+b)^{n+1} = (n+1)x^n,$$

which holds for any positive integer n as well as for $n = 0$, where as before b^i in the binomial expansion is to be replaced by b_i. In this relation we put x/q in lieu of x and obtain:

$$(x+q+qb)^{n+1} - (x+qb)^{n+1} = (n+1)qx^n.$$

In this relation we replace x by $x+q$, $x+2q \cdots$, $x+(k-1)q$, in turn and add the results. We obtain:

$$\frac{(x+kq+qb)^{n+1} - (x+qb)^{n+1}}{(n+1)q} = x^n + (x+q)^n + \cdots + \{x+(k+1)q\}^n.$$

This formula obviously gives the summation of the n-th powers of any set of integers in arithmetical progression in terms of x, n, q, and the Bernoulli numbers.

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[1] P. Bachmann, *Niedere Zahlentheorie*, vol. 2, p. 22.

[2] Quarterly Journal of Pure and Applied Mathematics, vol. 31 (1900) pp. 193–99.

[3] Lucas, *Theorie des Nombres*, Paris (1891) p. 240.

For the case $x=2$, $q=2$, as well as the case $x=1$, $q=2$, the formula was derived by Bachmann[4] by the use of two different expansions of

$$\sin nx \cos x/\sin x.$$

III. A CONVENIENT CHECK ON THE ACCURACY OF THE PRODUCT OF TWO MATRICES

By WILLIAM E. ROTH, Madison, Wisconsin

Let $A=(a_{ij})$ be a square matrix of order n, and designate the sum of the elements of the i-th row of A by a_i and the sum of the elements of the j-th column of A by α_j. Then

$$a_i = \sum_{j=1}^{n} a_{ij}, \quad \alpha_j = \sum_{i=1}^{n} a_{ij}, \quad (i,j = 1,2,\cdots,n). \tag{1}$$

Now we introduce the notation, $S(A)$, to designate the sum of all the elements of A; then

$$S(A) = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} \alpha_j, \tag{2}$$

according as we sum the elements of A by rows at a time or by columns at a time. With this definition of $S(\ \ )$ we can readily show that

$$S(A+B) = S(A) + S(B), \tag{3}$$

and, if m is a scalar multiplier of A, that

$$S(mA) = mS(A). \tag{4}$$

Now the element p_{ij} of the product $P=AB$ is given by

$$p_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \qquad (i,j = 1,2,\cdots,n),$$

and if we let the sum of the elements of the i-th row of B be given by b_i and that of the j-th column of B by β_j, to correspond to the notation employed for A, then

$$S(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} \alpha_k b_k. \tag{5}$$

Similarly

$$S(BA) = \sum_{k=1}^{n} \beta_k a_k.$$

In particular, $S(AB)=S(BA)$, when A and B are commutative; however this property is not a necessary one in order that the equality hold. From (5) we have at once

$$S(A^2) = \sum_{k=1}^{n} \alpha_k a_k. \tag{6}$$

[4] Loc., cit., pp. 292, 293.

If R is the inverse of A, and if r_i and ρ_j are the sums of the elements of its i-th row and of the j-th column respectively, then

$$(7) \qquad S(AR) = S(RA) = S(I) = \sum_{k=1}^{n} \alpha_k r_k = \sum_{k=1}^{n} a_k \rho_k = n.$$

Formulae (5), (6), and (7) are very convenient, though not absolute, checks on the accuracy of the products involved.

The idea above may be extended to products of three matrices and the extension is of interest but is not convenient for checking the accuracy of such products. The element p_{ij} of the product $P = ABC$ is given by

$$p_{ij} = \sum_{h=1}^{n} \sum_{k=1}^{n} a_{ih} b_{hk} c_{kj};$$

and if c_i and γ_j denote the sums of the elements of the i-th row and of the j-th column, respectively, of C, then we find that

$$S(ABC) = \sum_{h=1}^{n} \sum_{k=1}^{n} b_{hk} \alpha_h c_k.$$

This is a bilinear form whose matrix is B, the middle factor of the product. For A^r, where r is a positive integer, we have at once

$$S(A^r) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{r-2} \alpha_i a_j, \qquad (r \geqq 2),$$

where $a_{ij}^{(r-2)}$ is the element in the i-th row and the j-th column of A^{r-2}. This formula reduces to (6) if we agree to designate A^0 by I, the unit matrix. By virtue of (3) and (4) the operation here studied may be extended to polynominals in matrices. Special forms such as symmetric and skew-symmetric matrices have interesting properties when regarded in light of the above.

RECENT PUBLICATIONS

Edited by Roger A. Johnson, Hunter College, New York, N. Y., to whom books and communications should be sent.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

A Debate on the Theory of Relativity. By R. D. Carmichael, W. D. MacMillan, M. E. Hufford, and H. T. Davis. The Open Court Publishing Co., 1927. 154 pages.

The volume under review contains a report of the debate on the theory of relativity held at Indiana University in May, 1926 under the auspices of the

Indiana chapter of Sigma Xi. The most considerable direct attempt made by any of the speakers to define what a theory is or what properties a theory should have is contained in Professor Carmichael's opening speech. He points out (p. 36) that a theory "must be in suitable agreement with the facts of nature, it must have those aesthetic qualities which render it pleasing to the human spirit, and it must furnish what is to us the most agreeable theory from the point of view of convenience."

It seems advisable to state here, in somewhat explicit form, a view of the nature of a physical theory.[1] The statement is made because only in this way does it seem possible to estimate what this debate accomplished and what it did not. The remarks will refer to an ideal[2] rather than to any actual physical theory; and will be so brief as to contain, of necessity, certain crudities and inaccuracies.

An ideal physical theory starts with a set of undefined elements and certain primary unproved propositions concerning these elements. By the use of the canons of logic, there is then deduced a further set of propositions concerning the elements. These deduced propositions may be called the secondary propositions or the theorems of the theory. The undefined elements are then identified with certain physical entities, part of which at least are experimentally measurable. The secondary propositions or theorems thereby become proposed laws of nature, since they state quantitative relationships of uniformity in the external world which may be tested by experiment. The "theory" is then this entire logical structure,—undefined elements, undefined or primary propositions, the deduced theorems, and the proposed laws of nature which result from the theorem due to the physical identification of the elements.

One may then note three qualities which a theory may or may not possess. It may or may not possess the quality of *self-consistency*, according as the primary propositions are or are not consistent and according as there has or has not been an error in the purely logical structure of the theory.

Secondly, the theory may or may not possess the property of *correctness* or *truth*, according as the proposed natural laws do or do not turn out to be actual natural laws. It is doubtful whether the validity of a law can ever be rigorously established by experiment. Experimental evidence has, however, shown that some proposed laws must be discarded, and it has permitted others to be (temporarily, at least) retained. There will, moreover, always be border-line cases concerning which scientific men will disagree; and this disagreement is an inevitable and desirable result of the subjective judgments involved.

Thirdly, the theory may or may not possess the property of *value*. A little thought will convince anyone that a theory may be self-consistent and true,

[1] This statement is drawn, in part, from the penetrating analysis found in Chapter VI of N. R. Campbell's *Physics, The Elements*, Cambridge University Press (1920).

[2] Without this qualification the statement would give the false impression that actual physical theories have cleaned house, logically speaking, to the same extent as have certain branches of pure mathematics.

in the senses just discussed, but still be utterly valueless. The value of the physical theory seems to lie in its ability to "explain" the natural laws which are associated with it, and which are its fruition. This "explanation" is chiefly of two sorts, explanation by simplification and explanation by analogy; and the theory is, in the case of these two types of explanation, likely to be called a "mathematical theory" or a "mechanical theory" respectively.

The value which attaches to a theory which explains laws by simplification seems to arise from the subjective sense of satisfaction with which one recognizes that the several laws to which the theory leads appear as united and hence simplified through the central logical structure of the theory. In the case of explanation by analogy, the value of the explanation results partially from the same, and partially from other no less subjective considerations. There are important instances of theories in which there is possible a double identification of the undefined elements. One of these identifications leads to the theory itself, and the other to the analogy. Thus in the kinetic theory of gases certain elements are identified with the molecules of the gas and these elements are also thought of as (that is, identified with) minute perfectly elastic spheres. This second identification furnishes a mechanical model, and the double theory, considered as a whole, is a mechanical theory which explains by analogy. The satisfaction that any one person feels as a result of such an explanation is clearly a subjective matter. A great school of physicists has enjoyed mechanical theories and their explanations by analogy. The tendencies of modern theoretical physics have convinced a goodly number of persons that we should be very skeptical of most explanations by analogy. Thus, one who really believes in the electrical constitution of matter recognizes that the laws of ordinary mechanics are statistical consequences of the laws for individual electrodynamic action; so that a mechanical explanation of electrodynamics is without meaning or importance.

These remarks concerning the nature of a physical theory lead to three conclusions, each important for our present purpose. First, unless one is raising very fundamental questions concerning the validity of the canons of logic, there is a reasonably definite way of determining whether or not a theory is self-consistent.[3] Secondly, the question as to the truth of a theory must be attacked through experiment. After the experimental evidence is obtained, each person must decide for himself as to whether this evidence convinces him that the laws of the theory have or have not been established, this judgment being of an obviously subjective nature. Thirdly, the question as to whether a theory is valuable is, to a still greater extent, a subjective matter. It depends upon one's personal decision as to whether or not the theory explains its laws. Thus a theory can be true for one and untrue for another, and it may certainly be valuable to one and of little value to another. If one judges a theory to be valuable, he is the more likely to also judge it true, and vice

[3] The self-consistency of the theory of relativity was not questioned by any one of the debaters, although it has been seriously questioned by others.

versa. It has even been seriously suggested that some theories are doubtless valuable even though we judge them to be definitely untrue.

These remarks seem to the reviewer to make clear what such a debate could and could not accomplish, and they make it easy to report what actually did occur. In support of the theory, Professor Carmichael, opening for the affirmative, explained in an exceedingly clear and interesting way the nature of the restricted and the general theory of relativity. Only penetrating familiarity and long experience with the doctrines of relativity could produce so illuminating a review of its most important features. He called attention to the naturalness of the ideas, to the diverse laws which are obtained without the introduction of *ad hoc* hypotheses, and to the primitive simplicity of a space-time event as compared with the artificial separation into space and time. In the light of the remarks above, we should thus say that the first speaker expounded the theory and expressed and justified his subjective judgment that it has great value. He also affirmed his belief that the theory is true, but left arguments on this point to the other member of the affirmative team.

Professor MacMillan, opening for the negative, confessed that he is relatively indifferent as to whether or not the theory is true because, in his estimation, it is without value. It lacks value to him because its explanations do not explain. His intuitions are outraged by the abandonment of Euclidean space and Newtonian time, and the result is to him confusion rather than simplification. As one of the great thinkers on cosmological problems, he has on many happy occasions helped to enlarge our vision of those vast stretches of time during which our solar system has been evolving, and such considerations have helped him to gain a sort of cosmic patience. He is willing that science wait a few score or a few hundred years, if necessary, for a theory which will meet all our new experimental facts, but which will, at the same time, hold to the "postulates of normal intuition." He is unwilling to argue about the extremely delicate experimental tests because, true or not true, the theory makes him scientifically unhappy. It is a valuable thing that sound and sincere ideas of this sort be expressed. To seek to modify them through argument is idle and improper. The only way to change these opinions is to make Professor MacMillan over into some other person, and that is a project which American mathematicians will not support.

Following these two opening addresses, Professor M. E. Hufford, for the negative, and Professor H. T. Davis, for the affirmative, considered the experimental evidence for the truth of the theory of relativity. The Kaufman-Bucherer experiment, the Trouton-Noble experiment, the Michelson-Morley-Miller experiment, the Michelson-Gale experiment, the star deflection test, the spectrum shift test, the advance in perihelion test, and the evidence from spectral line structure are all discussed, pro and con. These delicate and subtle experiments cannot, in the reviewer's opinion, be profitably discussed before a popular audience and in an hour's time. It is not scientifically sound to represent that a discrepancy of one per cent or twenty per cent, or even fifty per cent

between theory and experiment is decisive evidence for or against a theory; nor is it safe to make the somewhat naive assumption that the minute one begins to talk of experimental evidence he is on "solid ground." Only those two great winnowers for the truth, time and the theory of probability, will be able to ultimately convince scientists that the first, second, . . . order effects of the theory of relativity are or are not confirmed by experiment. Professors Hufford and Davis had a difficult if not a hopeless task. Their discussion is interesting but it is, of necessity, not very enlightening or convincing.

The debate closed with short rebuttals, by Professors MacMillan and Carmichael, in which are restated and amplified their estimates of the value of the theory and its probable philosophical influence. The book, as a whole, is interesting and stimulating. It contains a useful summary of a good bit of experimental evidence. It contains illuminating and suggestive confessions of scientific faith. But if one wants to know whether the theory of relativity is, for him, a true and a valuable theory, he must steep himself in its doctrines and then search his own mind.

Warren Weaver

Operational Methods in Mathematical Physics. By Harold Jeffreys. London, Cambridge University Press, 1927. vi+99 pages.

Invariants of Quadratic Differential Forms. By Oswald Veblen. London, Cambridge University Press, 1927. vi+102 pages.

These two works constitute numbers 23 and 24, respectively, of the important series of monographs known as the "Cambridge Tracts in Mathematics and Mathematical Physics," and they maintain the high standard set by the previous numbers of that series.

A good idea of the scope of Jeffreys's work is conveyed by the chapter headings which follow: Fundamental Notions; Complex Theory; Physical Applications; One Independent Variable; Wave Motion in One Dimension; Conduction of Heat in One Dimension; Problems with Spherical or Cylindrical Symmetry; Dispersion; Bessel Functions. The main result of the first section of the work is the derivation of Heaviside's expansion theorem which gives in a compact and useful form the solution of a system of linear differential equations with constant coefficients satisfying assigned initial conditions. Since this theorem is implicit, and almost explicit, in the fundamental work of Cauchy[1] on linear differential equations, the especial merit of the operational methods in this connection would seem to be their compactness rather than their fruitfulness. However, the author proceeds to a discussion of the dynamics of continuous systems, which are dominated by partial differential equations rather than ordinary, and here the power of the operational method cannot be denied.

[1] See the following articles by the reviewer: *The Cauchy-Heaviside expansion formula and the Boltzmann-Hopkinson principle of superposition.* Bulletin of the American Mathematical Society, vol. 33, (1927), pp. 81–89; *The duty of exposition with special reference to the Cauchy-Heaviside expansion theorem,* this Monthly, vol. 34 (1927), pp. 234–241.

The statement by the author (p. 53) that a "general proof that the results given by the operational method, when applied to the vibrations of continuous systems, are actually correct, has not yet been constructed" should act as a challenge to mathematicians. The book is concise, almost terse, and will not prove easy reading for anyone but an expert, but we feel that many a reader will return to consult it often with profit.

Veblen's book is a very important contribution and it merits a somewhat detailed analysis. In Chapter 1 the algebraic part of the theory is discussed and use is made of the generalized Kronecker symbol (called by the author Kronecker delta) to develop the theory of determinants. In Chapter 2 invariants and tensors are introduced and the reader must be prepared for a terminology which is as yet not the general one. An invariant is an *object* not changed by transformations of coordinates; thus a tensor is a particular invariant, namely one whose components are point functions; in the formulæ of transformation for the components of a tensor a certain integral power N of the Jacobian is permitted, and N is called the *weight* of the tensor. When $N=0$ we have the classical tensor, and when $N=1$ we have a tensor density. Similarly, relative scalars are introduced. In Chapter 3 the quadratic differential form $ds^2=g_{\alpha\beta}dx^\alpha dx^\beta$ is introduced and the Christoffel three-index symbols are defined; it is shown that these satisfy, under a transformation of coordinates from x to $\bar{x}$, the familiar equations,

$$\bar{\Gamma}^r_{pq} = \Gamma^s_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^p}\frac{\partial x^\beta}{\partial \bar{x}^q}\frac{\partial \bar{x}^r}{\partial x^s}+\frac{\partial \bar{x}^r}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial \bar{x}^p \partial \bar{x}^q},$$

and the $\Gamma_{pq}{}^r$ are called the components of the invariant of affine connection. The formulæ for covariant differentiation follow; particularly interesting is that for the covariant derivative of a relative scalar. Geodesic coordinates simplify very much the derivation of fundamental formulæ, particularly the Bianchi identity. In Chapter 4 Euclidean geometry, i.e. the geometry for which the curvature tensor vanishes, and Euclidean affine geometry, i.e. the geometry of an affine connection whose components vanish in a particular coordinate system, are discussed; associated tensors (usually called reciprocal) are introduced and an interesting formula for the area of a 2-cell is given. The familiar concepts of curl and divergence are generalized. Chapter 5 is devoted to the equivalence problem. This is the problem of determining, if possible, a transformation from n x's to N u's such that $g_{\alpha\beta}dx^\alpha dx^\beta=\bar{g}_{\lambda\rho}\,du^\lambda du^\rho$, where g_{rs} and $\bar{g}_{pq}$ are given functions of x and u respectively. An instance of importance is the problem of imbedding a given curved space in a Euclidean space of more dimensions. Chapter 6 deals with the affine geometry of paths, a theory developed by the author and his colleagues at Princeton. Good use is made of affine normal coordinate systems which may be regarded as the natural analogue of Riemann's geodesic coordinates. A generalization of covariant differentiation, known as affine extension, follows and affine normal tensors, which are the extensions of the affine connection, are discussed.

Such is the book; and even a casual reader may well judge that only a master and creator of the subject could present so much intelligibly in 102 pages. Having given it the unnecessary tribute of our admiration we must now, if only to escape being insipid, call attention to some matters we regard as lapses from grace. In the historical remarks to Chapter 2, the term tensor is attributed to Einstein (1916); it is really due to W. Voigt, *Die fundamentalen Eigenschaften der Krystalle* (1898). The curl of a covariant tensor of rank one is not a metrical concept nor one of affine connection, but rather of analysis situs, and so its derivation by means of covariant differentiation is unfortunate. Similarly for generalized divergence and curl and the Stokes'-Poincaré tensor. The introduction of two concepts of divergence is apt to be confusing; in physical applications the divergence of the product of a covariant tensor density by a contravariant vector is what appears. This product ρu^r is an alternating covariant tensor of rank $n-1$ and from this can be formed in the general geometry of analysis situs, an alternating covariant tensor of rank n (a scalar density), namely, $(\partial/\partial x^\alpha)(\rho u^\alpha)$. The absolute scalar formed from this by scalar multiplication by the alternating contravariant tensor of rank n, $\pm g^{-1/2}$ is called the divergence. It is not the divergence of the contravariant vector u^r but of the contravariant vector density ρu^r. The only place where the metrical character of the space appears at all is in the step from the *relative* scalar $(\partial/\partial x^\alpha)(\rho u^\alpha)$ to the *absolute* scalar $g^{-1/2}(\partial/\partial x^\alpha)(\rho u^\alpha)$ or in replacing ρ by $\mu g^{1/2}$, where μ is an *absolute* scalar. The point raised here is rather essential since it concerns the foundations of electromagnetic theory. Maxwell's equations concern themselves with the ideas of curl and divergence; as such they have no concern with the metrical character of the space. It is only when we wish to generalize properly the equations $\mathbf{D}=\mathbf{E}, \mathbf{B}=\mathbf{H}$ that the metrics play any rôle.

F. D. Murnaghan

Non-Riemannian Geometry. By L. P. Eisenhart. American Mathematical Society Colloquium Publications, Volume VIII. 184+viii pp.

This book contains the material presented by Professor Eisenhart to the American Mathematical Society at its Ithaca Colloquium, September 1925, under the title, "The New Differential Geometry." The choice of subject-matter is well made from the standpoint of both the general reader and the specialist; and the exposition is in the author's usual lucid compact style.

The book divides itself into four chapters. Chapters I and II deal with affinely connected manifolds, the former with asymmetric, the latter with symmetric connections. The definition of an affinely connected manifold is an n-dimensional space to which a set of functions L_{jk}^{i} of the coordinates x has been assigned so as to define the parallel transportation of a contravariant vector ξ^i along an arbitrary curve by means of the differential equations

$$\frac{d\xi^i}{dt} = L_{jk}^{\ i}\frac{dx^i}{dt}\xi^k,$$

where t is the parameter on the curve. In I, no assumption of symmetry in j, k is made upon the n^3 functions L^i_{jk}, whereas in II this assumption is made and the components of the affine connection are then denoted by Γ^i_{jk}.

An interesting distinction, brought out in I, §12, between the two types of connection is that it is possible to change an asymmetric connection $(L^i_{jk}=L^i_{jk}+2\delta^i_j\psi_k)$ so as to leave undisturbed the relation of parallelism *in direction* between any two vectors with respect to any fixed curve joining their origins; but such change is not possible if we restrict ourselves to symmetric connections. A semi-symmetric connection (Schouten) (§14) is one relative to which there exists a symmetric connection with the same definition of parallelism in direction.

The paths of an affinely connected manifold are the curves whose tangent vectors are parallel (equipollent) with respect to the connection of the manifold; they are defined by the differential equation

$$\frac{d^2x^i}{dt^2} = \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}. \tag{1}$$

These paths appear each with a definite parameter determined up to linear transformation (affine parameter). We may take the paths, each with an affine parameter given upon it, as fundamental, and then a corresponding symmetric affine connection will be uniquely determined. To make the paths, as defined by (1), instead of the affine connection, the primary element of consideration is the spirit of the *geometry of paths* originated by Eisenhart and Veblen.

Other topics treated in Chapter II are normal coordinates; the extended theorem of Fermi, to the effect that by proper choice of a coordinate system we can make $\Gamma^i_{jk}=0$ all along a prescribed curve; extension of a tensor; and the equivalence problem for symmetric connections.

If the Γ's are transformed by the equation

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\phi_k + \delta^i_k\phi_j, \tag{2}$$

where ϕ_k is an arbitrary covariant vector, the paths remain the same although their affine parameterization changes. This is the basic fact of Chapter III, entitled, "Projective Geometry of Paths," which is the study of those properties of the paths that are independent of the particular affine connection used to generate them, or which are independent of any particular parameterization. At the end of this chapter the possible affine and projective collineations of a space of paths into itself are studied. An affine collineation is a point transformation which preserves the affine connection and therefore also the paths with their affine parameter; a projective collineation is required merely to preserve the paths and generally changes the affine connection according to (2), and correspondingly changes the affine parameter.

Chapter IV is devoted to the geometry of sub-spaces of the fundamental n-dimensional affinely connected manifold, the first part to hypersurfaces, of

dimensionality $n-1$, the latter part to sub-spaces of any dimensionality, $m<n$. An affine geometry is induced in a hypersurface of an affinely connected manifold as soon as we choose arbitrarily at each point of the hypersurface a contravariant vector, called the pseudonormal, to play a rôle like that of the unit normal to a hypersurface in a Riemannian space. This pseudonormal chosen, a geometry quite like that of a hypersurface embedded in a Riemannian space ensues. Thus we have asymptotic lines; conjugate directions; lines of curvature; an analogue to the geodesic curvature of a curve on a surface; generalized Gauss-Codazzi equations, which here as in the classic theory of surfaces appear as integrability conditions of a certain system of covariant partial differential equations; etc.

The above sketch may serve as a somewhat rough indication of what the book contains; for more definite ideas and details the text itself must be consulted. As appendix there is given an extensive bibliography arranged in chronological order, which will help the reader in furthering his acquaintance with the subject.

The proof-reading appears to have been especially well-done; in several readings of the text the reviewer did not notice any typographical error.

J. Douglas

Plane Trigonometry. By Harding and Mullins. The Macmillan Company, 1928. 118 pages.

What, another textbook on trigonometry? Yes, and one would almost say there is no need of any more since there are already too many. Then, why this one? Probably this question is best answered by the authors in their preface to this new book: "This text contains the material that is usually required in a semester's course in trigonometry. Since the present tendency is in the direction of a concise treatment, an effort has been made to present clearly and without too much detail the essentials of the subject."

The material is the same as that which is included in most texts on trigonometry. The arrangement of this material and the concise treatment of it are the distinctive features of the book.

The book begins with an introductory chapter on theorems in plane geometry which are most useful in the development of trigonometry. The authors omit, however, two theorems which are used later, the one that acute angles are equal if their sides are perpendicular, and the one that an inscribed angle in a circle is measured by one-half its intercepted arc.

Chapter II is a chapter on coordinates. The presentation is concise but clear.

Chapter III takes up "Angles and Trigonometric Functions." After an angle of any magnitude has been defined, the definitions of the trigonometric functions are given for an angle terminating in any quadrant. This makes for a clear understanding of the subject.

Chapter IV deals with the solution of the right triangle. The natural trigonometric tables are explained and then the tables are used in solving right triangles.

Right triangles are also solved by the use of logarithms but it is evident that the authors presuppose a knowledge of logarithms on the part of the student as no explanation of this theory is given.

In Chapter V come the various laws used in the solution of oblique triangles; and following these the solution of oblique triangles is taken up. This arrangement of the solution of oblique triangles immediately following the solution of right triangles before taking up the addition and subtraction formulas is made possible by the material given in the introductory chapter.

These first five chapters, consisting of 62 pages, cover all the work in so-called "Numerical Trigonometry." The remaining six chapters cover the work usually given in "Trigonometric Analysis."

In Chapter VI, "Reduction Formulas", the functions of angles of the second, third, and fourth quadrants are reduced to functions of a first quadrant angle in two ways. An angle of the second quadrant is thought of as being equal to 90° plus an angle of the first quadrant and also as 180° minus an angle of the first quadrant. Then follow the general laws of reduction.

Chapter VII is entitled "Relations Between the Functions." It first derives the fundamental formulas and then applies these formulas to the proof of identities and the solution of equations. The work in trigonometric identities is brief, almost too brief.

Chapter VIII, "Trigonometric Analysis," takes up the addition and subtraction formulas, double-angle formulas, half-angle formulas, and sum and difference formulas.

Chapter IX, "Unit of Measure," introduces the student for the first time to radian measure and applies it to the length of arc, and to angular and linear velocity.

Chapter X, "Inverse Trigonometric Functions," and Chapter XI, "Line Values and Graphical Representation," are the two final chapters.

The authors succeed in their effort "to present clearly and without too much detail the essentials of the subject." The treatment is concise but clear. It is not merely brief. The authors do not omit essentials nor do they put important material that should be explained in the text as problems to be worked out by the student. The flexibility of the text is such that a teacher desiring a short course covering the minimum essentials can omit the last three chapters without loss of continuity.

The book is a very readable one with neat appearing pages and many well labeled figures. The book contains a large number of carefully chosen problems illustrating the various topics. The problems illustrating the right triangle and oblique triangle are especially well-chosen.

T. E. Mergendahl

College Algebra. By C. I. Palmer and W. L. Miser. New York, McGraw-Hill Book Company, Inc., 1928. 377 pages, including answers. $2.50.

College Algebra by Palmer and Miser is conventional in its development and

treatment of the subject. The material has been selected and arranged, however, so that the text can easily be adapted to classes that vary considerably in preparation and for courses that differ in extent from one of three hours to one of five hours.

There is a wealth of well-graded exercises and problems, a large number of the latter being of a practical nature. The paragraphs on "equivalency" should be suggestive to the better grade of student. The chapter on "inequalities" includes work that will be of assistance in the discussion of equations and the plotting of their loci in Analytics.

In the preface, the authors state: "In the development of ideas and in the discussions, the average student has been kept constantly in mind; and it is to him the authors have addressed themselves rather than to the trained mathematician." The average student certainly ought to be able to read this text with understanding.

The publishers have put out the book in attractive form: the paper, type, and figures are all good.

On the whole, it appears to be a teachable text and one that ought to be very satisfactory.

James E. Donahue

MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

The Mathematics Section of the Science Club of Hanover College, Hanover, Indiana.

The officers for 1927–28, first semester, were: Frank Bard, president; Margot Lambertson, secretary; second semester, Jesse Harmon, president; Mary E. Holderman, secretary.

Officers for 1928–29, first semester, are George Bishop, president; Louise Plummer, secretary; Professor H. A. Zinszer, faculty adviser.

The program for the year 1927–28 included the following papers:

October 12, 1927. "Trigonometric functions," by Margaret Bellamy, '29.

November 9. "The function concept in geometry," by Eugene Crouch, '28; "The Pythagorean theorem," by Hope Rankin, '29.

December 14. "Matter, space, and time," by Professor H. T. Davis (Indiana University).

February 8, 1928. "Fresnel's mathematical theory of diffraction," by Professor Harvey A. Zinszer.

March 14. "The mathematics of chemistry," by George Balas, '28.

April 11. "Cubic equations," by Harland Harris, '28.

May 9. Picnic at Butler Falls.

May 23. "Summation of series," by Raymond Park, '28.

The club meets semi-monthly; one meeting is sponsored by the mathematics and physics sections, the other by the chemistry, biology and geology sections. The club sent four of its members as delegates to the state mathematical meeting at Butler University.

(Report by Miss Louise Plummer)

Mathematics Club of Washington Square College, Evening Organization, Washington Square, New York.

Professor Frederick John, Adviser.

The program of the Washington Square College Mathematics Club for the year 1927–1928 was the following:

October 19, 1927. Plans for year discussed.

November 16. "Determination of π," by Mr. Henry Salkind, '29.
"Squaring a circle," by Mr. Henry Salkind, '29.

December 14. "Zeno's paradoxes," by Mr. Louis Kalinkowitz, '29.

January 4, 1928. "Perpetual motion," by Mr. Henry Salkind, '29. "Mathematical fallacies," by Miss Theresa Zwerling, '30.

February 29. "Proofs of the Pythagorean theorem," by Mr. William Schutzman, '30.

March 28. "Life of Sir Isaac Newton," by Mr. Nathan Sirota, '30.

April 11. "Relativity," by Mr. Charles K. Payne, member of the faculty.

April 25. "Relativity," (concluding lecture) by Mr. Charles K. Payne, member of the faculty.

(Report by Morris Kline, Chairman)

The Undergraduate Mathematics Club of the State University of Iowa.

The officers for the year 1927–1928 were: Dr. N. B. Conkwright, faculty adviser; H. A. Meyer, president; Dorothy McCoy, secretary.

October 20, 1927. "Some remarkable points of the plane triangle," by Professor Roscoe Woods.

November 3. "Problem of packing," by Professor R. P. Baker.

November 17. "Isotropic lines," by Professor L. E. Ward.

December 1. "Cissoids," by Mr. Watson Davis.

January 12, 1928. "Logarithms of complex numbers," by Mr. C. S. Carlson.

February 9. "Some astronomical magnitudes," by Mr. C. C. Sherman.

February 23. "Trends of present day mathematics," by Mr. S. H. Huffman.

March 8. "Mathematical recreations," by Mr. Fred Webber.

March 22. "Sir Isaac Newton and contemporary mathematicians," by Mr. H. A. Wright.

April 19. "Convergence tests," by Dr. N. B. Conkwright.

(Report by Miss Dorothy McCoy, Secretary)

The Mathematics Club of Mount Holyoke College, South Hadley, Massachusetts.

The following programs were given in the year 1927–28:

October 15, 1927. "Summer work with the Western Electric Company," by Margaret Crierson, '29. "Foreign coins," by Stephanie Locke, '28.

November 12. "Maps," by Professor James Pierpont of Yale University.

January 21, 1928. "Fiedler's cyclography," by Professor B. H. Brown, of Dartmouth College.

February 18. "Christopher Rudolph's *Die Casse*," by Margaret Weeber, '28. "Stifel's *Arithmetica Integra*, by Mabel Miller, '28. "Early mathematical texts in the United States," by Stephanie Locke, '28.

March 9. "Explanation of models of algebraic surfaces," by Professor Emilie N. Martin, of Mount Holyoke College. "Blaise Pascal," by Harriet Smith, '28.

April 7. "The logistic curve," by Professor Lowell J. Reed, of Johns Hopkins University.

The Mathematics Club of the University of Kansas, Lawrence, Kansas.

The officers of the Mathematics Club of the University of Kansas for the year 1927–28 were: President, Leslie McKeehen, '28; Vice-president, Byron Rexroth, '28; Secretary-treasurer, Winnona Venard, '28.

The following topics were presented at the meetings:

October 3, 1927. Business meeting.

October 17. "Mathematics in Italy," by Professor E. B. Stouffer.
November 7. "Pythagorean numbers," by Mabel Penrod, Gr.
November 21. "Life of Leibnitz," by Mary Bates, '28. "Figurate numbers," by J. P. Jenison, Gr.
December 5. "Trilinear coordinates," by Corinne Hattan, Gr.
January 6, 1928. "Clocks and timepieces," by Edwin Titt, Gr.
February 6. "Fundamental notions of relativity," by Professor J. J. Wheeler.
February 20. "Curve fitting," by James Edson, '28.
March 5. "Mexico," by J. M. Gonzalez, '28. "Codes and ciphers," by Ruth Swonger '29.
March 19. "Linkages," by George Heald, Gr. "Newton's pasturage problem," by Florence McClure, '29.
April 2. "Non-Euclidean geometry," by Josephine Braucher, Gr.
April 16. "Russian peasant method of multiplication," by Helen Trotter, '29. "Card tricks," by Edna Dobson, '29.
May 7. "Practical applications of complex numbers," by Millard Smith, '28.
May 14. Mathematics Club Annual Picnic.

(Report by Winnona Venard)

Newtonian Society of the State College of Washington, Pullman, Washington.

The program of the Newtonian Society of the State College of Washington for the year 1927–28 was the following:

October 11, 1927. Election of officers for year 1927–28: President Miss Arlene Perry, '28: Secretary-treasurer, Miss Pansy Swannack, '28; Reporter, Mr. Clarence Ross, '29. "History of the Newtonian Society," by Florence Johnson, '27. "Conformal representation," by Professor C. A. Isaacs. "Mapping," by Miss Arlene Perry, '28.
October 26. "Perfect and amicable numbers," by Miss Pansy Swannack, '28. "Theory of numbers," by Professor E. C. Colpitts.
November 8. "Development of number systems," by Miss Audrey Graber, '28. "Significance of the theory of functions of a real variable," by Professor H. H. Irwin.
November 22. "Four harmonic points," by Miss Merna Pell, '30. "Projective geometry," by Professor J. R. Vatnsdal.
December 6. "A peculiar sort of algebra," by Mrs. Lorna Herman,'29. "The theory of groups," by Dr. S. E. Shelkunoff.
January 10, 1928. "Astronomy," by Professor C. D. Calogeris and Mr. Ross, '29.
January 24. "The great pyramid of Egypt," by Mrs. A. T. Mills and Miss Uarda Davis, '28.
February 21. "Classification of numbers," by Miss Marguerite Feix, '29. "Development of numbers," by Miss Florence Johnson, '28.
February 28. "Peano," by Mr. Leland Smith, '29. "Mathematical logic," by Professor C. A. Isaacs.
March 9. Annual Banquet.
March 20. "Chinese numbers," by Mr. Kwang Chang, '28. "Fun fest," by the Seniors.

(Report by Professor C. A. Isaacs)

The Mathematics Club of Columbia College, New York, N. Y.

The program of the Mathematics Club of Columbia College for the year 1927-1928 was the following:

October 25, 1927. Meeting for organization conducted by Professor Fite, adviser of the Club. At the request of the members, the faculty appointed the following committee to take charge of the affairs of the Club: Carl Boyer, '28, Chairman; Isidore Kagno, '30; and Aaron B. Muravchek, '30.
November 9. "The trisection of an angle," by Mr. Abraham Gansler, '28.
December 1. "Transfinite numbers," by Mr. H. W. Raudenbush.
December 15. Informal discussion on topics of interest to the members.

January 12, 1928. "Elimination," by Mr. Samuel Borofsky.
February 14. "The theory of substitutions," by Professor Siceloff.
February 28. "Mascheroni constructions," by Mr. Martin Rosenman, '28.
March 14. "Geometry," by Professor Pfeiffer.
March 21. Continuation of the address by Professor Pfeiffer.
April 3. "Interpretation of the Euclidean postulates," by Mr. Edgar R. Lorch.
April 26. "The theory of equations," by Mr. Herbert Hinman, '28.
May 9. "The significance of existence theorems," by Professor Fite.

(Report by Carl Boyer, '28, Chairman of the Executive Committee)

The Mathematics Club of Brown University, Providence, R. I.

The following program is to be given during 1928–1929:
October 30, 1928. "Neighboring worlds," by Gerald Maurice Clemence, '29. "The astrologer's quest," by Myrtle Congdon Ryder, '31.
November 20. "Soap bubbles," by Mabel Louise Blaney, '29. "The modern mariner," by Charles Henry Vehse, Gr.
January 22. "Some geometrical maxima," by Professor Jacob David Tamarkin, of Brown University.
February 26, 1929. "The game of nim and its theory," by Helen Anna Sparrow, '30. "Skeleton division, magic squares," by David Joseph Colbert, '29.
March 19. "The four color problem," by Allen Fuller Pomeroy, '29. "One-sided surfaces," by Edward Mason Read, 3rd, '31.
April 23. "Maps," by Professor James Pierpont, of Yale University.
May. Picnic.
Committee on Program: Professor Bennett; Doctor Hickson; Mabel Louise Blaney, '29; Helen Anna Sparrow, '30; David Joseph Colbert, '29; Peter Shahdan, '30.
Committee on Arrangements: Charles Hill Wallace Sedgewick, Gr.; Mary Honor Cummings, '29; Myrtle Congdon Ryder, '31; Allen Fuller Pomeroy, '29; Edward Mason Read, 3rd, '31.

(Report by Professor R. C. Archibald).

The Mathematics Club of the College of the City of Detroit, Detroit, Michigan.

The Mathematics Club of the College of the City of Detroit held five meetings during the year 1927–1928 at which papers were presented by students and members of the faculty.
October, 1927. "The mathematical theory of statistical correlation," by Mr. William Borgman.
November 23. "The applications of the theory of equations to organic chemistry," by Mr. Donald Murphy.
January 18, 1928. "Line values of trigonometric functions, and their use in constructing curves," by Miss Jean Persons.
March 20. "Magic squares," by Miss Evelyn Raney.
April 17. "Quadrature of the circle, History of π," by Miss Lura Green.

Officers for the first half of the year were: Mr. W. Herbert Bixby, President; Miss Elinor Batie, Secretary.

For the last half of the year officers were: Miss Evelyn Raney, President; Miss Elinor Batie, Secretary.

(Report by Elinor M. Batie, Secretary)

The Mathematics Club of Cooper Union, New York, N. Y.

The officers for the year 1927–1928 were: Peter Douglas, '28, president; James McConaghy, '30, vice-president; Edward Trapani, '29, secretary-treasurer. Meetings were held at intervals of about three weeks throughout the year with the following program:
November 9, 1927. "Curious properties of numbers," by A. H. Beiler, '25, of the American Gas and Electric Company.

November 30. "Mechanical paradoxes," by Abraham Berkowitz, '28.
December 21. "Stadia surveying," by Kaskel Kallman, '31.
January 11, 1928. "Infinity," by W. G. Findley, Instructor in the Department of Psychology.
February 1. Visit to the Museum of Peaceful Arts. Explanation of mathematical and scientific instruments by Edwin Schwarz, '27.
February 29. "Mental calculations," by Abraham Rosenbaum, '29.
March 31. Joint meeting of the student branches of the A. S. C. E., A. S. M. E., A. I. E. E., the Mathematics Club, and the Chemistry Club. "Human relations," by W. Chevalier, general manager of the Engineering News-Record.
April 11. "Projective geometry," by Professor H. W. Reddick. Election of officers.

A copy of Ball's *Mathematical Recreations and Essays* was awarded by the Club to James J. Murphy, '31, for excellence in mathematics in the first year class.

(Report by Professor H. W. Reddick)

The Mathematical Society of Rutgers University, New Brunswick, N. J.

The officers for 1927–28 were: President, Irving H. Worden; Vice-president, Joseph Ensley Clayton; Secretary-treasurer, Charles R. Eason. Regular monthly meetings, besides two joint meetings with the Society at New Jersey College for Women, were held and papers were read as follows:

Professor S. E. Brasefield, "Harmonic motions."
Professor E. P. Starke, "Trilinear coordinates."
Professor C. M. Huber, "Mathematics of æronautics."
Professor W. E. Breazeale, "The path to the house of light."
Professor Richard Morris, "Two circles of the latter part of the 19th century."
Mr. H. H. Pixley, "Exponentials."
Mr. Chas. R. Wilson, "The logarithms of negative numbers."

Student Papers

Irving H. Worden: "A geometric construction."
Reuben McDaniel: "Theory of congruences" and "Functions of multiple angles."
J. E. Clayton: "Concurrent diagonals of a quadrilateral."
Charles Eason: "Continued fractions."
R. J. Miejdak: "Indeterminate equations."
Charles Higgins: "The Delian problem and the trisection of an angle" and "Vandermonde's theorem."
N. C. Giordano: "A geometrical construction."
Jacob Neuss: "Theorems of Menelaus and Ceva."
Edward Green: "Inverting a series."
C. P. Booraem: "Summations of series by the method of differences."

As an appropriate ending to a very successful year, there was an attendance of twenty two at the annual banquet in May.

(Report by Professor Richard Morris)

The Mathematical Society of the New Jersey College for Women, Rutgers University, New Brunswick, N. J.

During the year 1927–28 the regular meetings were changed from monthly to twice monthly; the usual social events of a reception for new members, a Christmas party, the annual banquet, a subscription dance, a farewell to the seniors, and a picnic in May tendered by Professor and Mrs. C. A. Nelson were given. At a joint meeting with the Men's Club from the University, Professor Solomon Lefschetz of Princeton University gave a lecture on analysis situs. At this meeting there was present a number of teachers from high schools nearby.

At the regular meetings, a member of the faculty had a main topic and he associated with him members of the student body who were interested in related topics.

November 15, 1927. Professor A. A. Titsworth: "Linear perspective." Florence Buckley and Gladys Francis: "The relation of linear perspective to art."

December 13. Professor Richard Morris: "Mid-points of segments as vertices of polygons." Josephine Pokorney: "History of geometry." Marguerite Heyer: "Proof of Desargues's theorem."

January 8, 1928. Professor W. E. Breazeale: "Waves of light." Dora Thorpe: "Theories of light propagation." Leonora Gross: "The velocity of light."

February 14 and 28. Professor C. A. Nelson: "Quadratic equations." Ruth Horsefield and Ruth Donner: "Methods of solution." Margaret Banta, Anna Hobbs, and Osie Labaw: "Solutions of problems."

March 27. Professor Richard Morris, Ruth Bump, Margaret Frahme, and Helena Doane: "The nine-point circle and sums of certain ratios." H. H. Pixley: "Gauss's definitions." Edward A. Green: "A problem in analysis." Chas. P. Borraem: "Methods of differences."

April 24. A. E. Meder, Jr.: "Introductory remarks on order." Virginia Young, Gladys Francis, Mildred Bennett, and Gertrude Gard: "Types of serial order."

May 8. Helen Haerter: "Definition of determinants and development by minors." Ruth Nixon: "Properties of determinants." Katherine Roelker: "Evaluation by minors." Ruth Thomspon: "Summary."

(Report by Professor Richard Morris)

PROBLEMS AND SOLUTIONS

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

SOLUTIONS

3089 [1924, 353]. *Proposed by Norman Anning, University of Michigan.*

Given four points, O, A, B, C, on a straight line, to construct with straight edge only, the point P on the line such that OP shall be the harmonic mean of OA, OB, OC.

Solution by Otto Dunkel, Washington University.

It will be convenient to denote the four points on the straight line by the letters O, U, V, W; it is desired to locate a point P on the same straight line so that $3/OP = 1/OU + 1/OV + 1/OW$.

Through U, V, W draw the sides of a triangle BC, CA, AB, and let AO, BO, CO cut the opposite sides of this triangle in A', B', C' thus forming a new triangle $A'B'C'$. Let the corresponding sides of the two triangles meet in C'', A'', B''. Then by Desargues's theorem A'', B'', C'' lie in a straight line. It will be shown that the intersection of this line with the given line is the desired point P.

From the quadrilateral $AC'OB'$ we see that $BA'CA''$ is an harmonic set; and the same is true of $AC'BC''$ and $CB'AB''$, for similar reasons. We shall use the following theorem which is easily proved: Given any cubic and a fixed point O, let the variable secant line OP cut the cubic in the points U, V, W.

Then, if k is a constant, the locus of the point P such that $1/OU+1/OV+1/OW = k/OP$ is a straight line. If $k=3$ this locus is called the polar line of O with respect to the cubic. Regarding ABC as a cubic, we shall consider the polar line o of O with respect to this cubic. Let L be the intersection of o with BC, and draw the secant OL cutting CA in M and AB in N. Then $3/OL=1/OL+1/OM+1/ON$, or $2/OL=1/OM+1/ON$, and hence $OMLN$ is an harmonic set. But, since the pencil $A(MONA'')$ is harmonic, L must lie on AA'' as well as on BC. Therefore L and A'' coincide. It may be shown similarly that o goes through B'' and C'', and this completes the proof.

The polar conic of O may be determined by the fact that it must pass through the double points A, B, C, and by the general property that the polar of O with respect to the conic must be the polar line of O with respect to the cubic, that is $A''B''C''$. Now the polar of A'' with respect to the polar conic must be $A'O$, and thus $A''A$ is the tangent at A. Likewise $B''B$ is the tangent at B. Now these five conditions suffice to determine the conic, and this conic must necessarily be tangent to CC'' at C.

If we have five points O, U, V, W, X on a straight line, it is obvious that there is a straight edge construction for the point P such that OP is the harmonic mean of OU, OV, OW, OX.

3291 [1927, 491]. *Proposed by C. N. Mills, Normal, Illinois.*

Given a circle of radius R; AB the side of an inscribed octagon; ACE three consecutive points of the inscribed hexagon; AD the side of an inscribed square. BD intersects CE at O. With O as a center and the chord BC as a radius, describe a circle which intersects AD at P. Show that AP is the side of the inscribed heptagon, with an error of $.00105R$.

Solution by Theodore Bennett, University of Illinois.

By a suitable choice of rectangular coordinate axes the given points may be made to have the coordinates

$$A = (R,0),\ B = (\tfrac{1}{2}R\cdot 2^{1/2},\ \tfrac{1}{2}R\cdot 2^{1/2}),$$
$$C = (\tfrac{1}{2}R,\ \tfrac{1}{2}R\cdot 3^{1/2}),\ D = (0,R),\quad E = (-\tfrac{1}{2}R,\tfrac{1}{2}R\cdot 3^{1/2}).$$

Then we find that

$$O = [\tfrac{1}{2}R(2 + 2^{3/2} - 3^{1/2} - 6^{1/2}),\tfrac{1}{2}R\cdot 3^{1/2}] \quad \text{and}$$
$$BC = R[2 - (1 + 3^{1/2})\cdot 2^{-1/2}]^{1/2}.$$

Then the point P can be determined by the standard methods of analytic geometry. Eventually we find that

$$AP = \tfrac{1}{2}R[-2 + 3^{1/2} + 6^{1/2} - (1 - 2\cdot 2^{1/2} + 4\cdot 3^{1/2} - 2\cdot 6^{1/2})^{1/2}] = .866719R.$$

The side of the inscribed heptagon is $2R\sin(\pi/7) = .867767R$, which differs from AP by $.00105R$.

Also solved by C. J. Stowell.

3293 [1927, 491]. *Proposed by R. E. Gaines, University of Richmond.*

From the foci F and F' of a conic, lines FP and $F'P'$ are drawn parallel to each other and cutting the conic in P and P'; find the envelope of PP'.

Solution by Roscoe Woods, State University of Iowa.

It suffices to determine the envelope for the ellipse $b^2x^2+a^2y^2=a^2b^2$, for we have merely to replace b^2 by $-b^2$ in the resulting equation in order to obtain the result for the hyperbola. Instead of the foci two points $D(d,0)$ and $D'(-d, 0)$ will be used. The two parallel lines PDQ and $P'D'Q'$ have the equations $ny-x\pm d=0$, where n has any value; and we suppose that they cut the conic in P, Q, P', Q'. The lines PQ' and $P'Q$ always pass through the origin for all values of n, and do not therefore envelop a curve in the proper sense. The lines PP' and QQ' are parallel and determine with the given conic a system of conics through the four points P, Q, P', Q' having the equations

$$b^2x^2 + a^2y^2 - a^2b^2 + t(n^2y^2 - 2nxy + x^2 - d^2) = 0,$$

where t is the parameter of the system. Certain values of t determine the degenerate members of the system. It is easily found that $t=-a^2b^2/(a^2+n^2b^2)$ gives the member consisting of the pair of parallel lines PP' and QQ', and that the equations of these lines are

$$b^2nx + a^2y = \pm ab(a^2 - d^2 + b^2n^2)^{1/2}.$$

The envelope, which is found in the usual way, is a conic with the equation

$$b^2(a^2 - d^2)x^2 + a^4y^2 = a^2b^2(a^2 - d^2).$$

The case of $d=a$ will be disregarded as trivial. The envelope has the same x-intercepts, $\pm a$, as the original conic. If the original conic is an ellipse, the envelope is an ellipse or hyperbola according as $a>d$ or $a<d$; but if an hyperbola, the envelope is ellipse or hyperbola according as $a<d$ or $a>d$.

If $d^2=a^2-b^2$ the points D, D' become the foci, F, F' of the problem, and the above equation of the envelope is now $b^4x^2+a^4y^2=a^2b^4$, and the envelope is an ellipse. Thus both the ellipse and the hyperbola having the same value for a and the same absolute value for b yield the same envelope. The y-intercepts for the envelope are equal to the semi-latus rectum of the given conic. By finding the similar envelope for this new ellipse and repeating this process indefinitely, we determine an interesting system of ellipses through the points $(\pm a, 0)$.

A similar method could be used for the parabola by taking two points on the axis, but the lines forming the degenerate conics are not equally inclined to the axis, and the work of isolating the envelope is therefore more complicated.

Note by the Editors. The generalized form of the problem may be stated in a way which suggests a method of finding the equations of PP' and QQ' in a manner different from the above.

Let D be a fixed point on one of the principal axes of a conic and let PDQ

be a variable chord. Find the envelope of the chords through P and Q supplementary to PQ.

Also solved by E. F. Allen, William Hoover, H. R. Kingston, and the Proposer.

3295 [1927, 537]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Given a complete quadrilateral it is possible to describe three circles having their centers at the vertices of the diagonal triangle so that the vertices of the quadrilateral shall be the centers of similitude of the three circles taken in pairs. Moreover one of the three radii may be taken arbitrarily.

NOTE. This is the converse of the known proposition: The centers of similitude of three given circles taken in pairs are the vertices of a complete quadrilateral. See, for instance, Nathan Altshiller-Court, *College Geometry*, p. 160, Johnson Pub. Co., Richmond, Va., 1925.

Solution by A. Pelletier, Montreal, Canada

Let $ABCD-EF$ be a complete quadrilateral where AB and CD meet in F, BC and AD meet in E, AC and BD meet in O, and these last two lines meet FE in O'' and O'. Then $BODO'$, $AOCO''$, $FO''EO'$ are harmonic sets.

Let r (any value) be the radius of a circle (O) with center O, and take $r'=rBO'/BO$ as the radius of circle (O'), $r''=rAO''/AO$ as radius of (O''). Then B and D are centers of similitude for (O) and (O'), and A and C are similar centers for (O) and (O''). Menelaus' theorem applied to the triangle $OO'O''$ and the secant ABF gives the equality

$$(AO/AO'')(BO'/BO)(FO''/FO') = 1.$$

Hence $r'/r''=FO'/FO''$; and, since F and E divide $O'O''$ harmonically, they are centers of similitude for (O') and (O'').

3296 [1927, 537]. *Proposed by J. Rosenbaum, Milford, Connecticut.*

It is well known that the radius of the inscribed circle of a right triangle is equal to half the difference between the sum of the legs and the hypotenuse. Derive an analogous expression for the radius of the inscribed sphere of a right tetrahedron.

Solution by Otto J. Ramler, Catholic University, Washington, D. C.

Let the edges of the right trihedral angle of the right tetrahedron O-ABC be a, b, c. Let the areas of the triangles AOB, BOC, COA, ABC be A_z, A_x, A_y, A, respectively. Then it is well known that $A_x^2+A_y^2+A_z^2=A^2$. (See Osgood and Graustein, *Plane and Solid Analytic Geometry*, p. 517, Th. 2.)

If r is the radius of the inscribed sphere, we have

$$6\cdot\text{Volume } O\text{-}ABC = abc = 2r(A_x + A_y + A_z + A).$$

Solving for r, we have

$$r = \frac{abc}{2(A_x + A_y + A_z + A)} = \frac{abc(A_x + A_y + A_z - A)}{2[(A_x + A_y + A_z)^2 - A^2]} = \frac{A_x + A_y + A_z - A}{a + b + c},$$

when proper substitutions are made in the denominator.

Also solved by Elizabeth E. Nixon, A. Pelletier, G. A. Yanosik, and Paul Wernicke.

3297 [1927, 537]. *Proposed by C. N. Mills, Normal, Illinois.*

Suppose BOD to be a quadrant of a circle of radius R; find the radius of a circle inscribed therein. Also find the radius of a circle which will touch both circles and the line OB.

Solution by J. H. Neelley, Carnegie Institute of Technology.

The center Q of the inscribed circle must lie on the bisector of the angle BOD. Let OQ cut the given quadrant of a circle in P, and let C be the foot of the perpendicular from Q to OB. Then the radius of the inscribed circle is $x = OC = 2^{-1/2} \cdot OQ$. Since $OQ = R - x$, we have at once $x = (2^{1/2} - 1)R$.

The second part of the problem has the trivial solution of a circle of radius x inscribed in the second quadrant tangent to OB at C. This fact will be utilized later. Let the second required circle be tangent to OB at F and have center E, and let N be the foot of the perpendicular from E to CQ. Then, since OE passes through the point of tangency of this circle and the quadrant of a circle $OF^2 = R(R - 2y)$, where y is the radius of the circle. In the right triangle ENQ, $EN = OF - x$, $NQ = x - y$, $EQ = x + y$. Hence $(OF - x)^2 = (x + y)^2 - (x - y)^2 = 4xy$. Eliminating OF by means of the previous equation and inserting the value of x, we obtain the equation

$$[2^{3/2} - 1]^2 y^2 - [14(2)^{1/2} - 18]Ry + [2^{1/2} - 1]^2 R^2 = 0.$$

This gives, on rejecting the trivial solution, $(2^{1/2} - 1)R$,

$$y = \frac{(2^{1/2} - 1)R}{(2^{3/2} - 1)^2} = \frac{[5(2)^{1/2} - 1]R}{49}.$$

Also solved by E. C. Kennedy, H. M. Lufkin, Elizabeth E. Nixon, A. Pelletier, O. J. Ramler, and F. L. Wilmer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

During the first two years of its operation, the Information Bureau to which reference is made on page 1 of this issue, under the direction of Professor H. W. Kuhn, has received 115 registrations of candidates for appointment, and has answered inquiries from 18 institutions in search of candidates. It is hoped that the use of its services will be largely increased as the nature of its work becomes better known.

Professor Hermann Weyl, of Princeton University, delivered an address on the "Law of conservation and rules of intensity in quantum theory" at the University of Minnesota at the dedicatory exercises for the new physics building on November 30, 1928 in conjunction with the November meeting of the American Physical Society.

Professor W. B. Ford, of the University of Michigan, has been given leave of absence for the current academic year. He is traveling in Europe and, as representative of the "Carnegie Endowment for International Peace," he will lecture at the Universities of Leiden and Utrecht, the University of Brussels, the Universities of Lille and Grenoble, and the University of Pisa.

Dr. Charles G. Crooks, for twenty-eight years professor of mathematics in Centre College and for a number of years dean of the college, has retired as dean emeritus because of ill health.

Assistant Professor J. R. Kline has been promoted to a full professorship at the University of Pennsylvania.

Dr. John Williamson has been appointed associate in mathematics at Johns Hopkins University.

Announcement is made of the death of William Theodore Gauss on November 14, 1928, at his home in Colorado Springs, Colorado, at the age of seventy-seven years. Mr. Gauss was a grandson of the illustrious German mathematician, Carl Friedrich Gauss, and through his mother was a nephew of the noted German astronomer, Friedrich Wilhelm Bessel. For many years he spent much time and money in assembling a large and valuable collection of Gauss memorabilia, which is now being used by Mr. G. Waldo Dunnington, of Washington and Lee University, in a biography of C. F. Gauss. This death reduces the number of Gauss's grandchildren to four, one of whom resides in California; the others reside in Missouri. His brother, the Reverend Dr. Joseph Gauss, is superintendent of the Brooks Bible Institute in St. Louis, Missouri.

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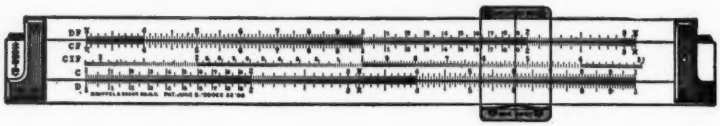

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The Association needs funds for scientific publications and for the promotion of scientific activities.

Missing Numbers of the Monthly

Cash, or credit toward future dues, will be given for certain single numbers as follows, up to a limited number of copies: February, March, May, September, 1913; September, 1914; February, March, April, June, 1915; February, September, 1918—fifty cents; September, 1915—seventy-five cents; May, 1915—one dollar. (See MONTHLY, March, 1921, p. 152); October, 1920; August-September, October, 1921; May, September, October, 1924; May, June-July, November, 1926—forty-five cents.

Address all communications to the

***Secretary*, W. D. CAIRNS**
OBERLIN, OHIO

CONTENTS

The Information Bureau for Appointments 1
The Mathematical Association of America. A List of New Members ... 1
The November Meeting of the Southern California Section. By P. H. DAUS 2
Graphical Intergration and Differentiation of Functions in a Polar Coordinate System. By E. A. KHOLODOVSKY 3
Heronian Triangles. By WILLIAM FITCH CHEYNEY, JR. 22
How Can Interest in Calculus be Increased? By ROSCOE WOODS 28
Generalizations of the Theorem of Pythagoras and Euclid's Theorem of the Gnomon. By WILLIAM J. HAZARD 32
QUESTIONS AND DISCUSSIONS: Discussions—"A model for showing sines and cosines," by WILLIAM J. HAZARD; "An extension of the Bernoulli summation formula," by H. S. VANDIVER; "A convenient check on the accuracy of the product of two matrices," by WILLIAM E. ROTH 35
RECENT PUBLICATIONS: Reviews by WARREN WEAVER, F. D. MURNAGHAN, J. DOUGLAS, T. E. MERGENDAHL, JAMES E. DONAHUE 38
MATHEMATICS CLUB: Club activities 48
PROBLEMS AND SOLUTIONS: Solutions—3089, 3291, 3293, 3295, 3296, 3297 53
NOTES AND NEWS 58

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August, 1929.

Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled.

ILLINOIS, Carthage, Ill., May 3-4.
INDIANA, Culver Military Academy, May 3-4.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN.
MINNESOTA, St. Paul, May.
MISSOURI, Kansas City, Mo., November.
NEBRASKA.
OHIO, Columbus, Ohio, April 4.
PHILADELPHIA.
ROCKY MOUNTAIN, Greeley, Colo., April.
SOUTHEASTERN.
SOUTHERN CALIFORNIA, University of Redlands, March 9.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

ON STERN'S DIATOMIC SERIES

By D. H. LEHMER, Brown University

The purpose of the present paper is to extend the investigation of the following "Diatomic Series" studied by Stern.[1]

0	1															1
1	1								2							1
2	1				3				2				3			1
3	1		4		3		5		2		5		3		4	1
4	1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
	.	.	.		.	.	.	.	.		.	.	.	.	.	.	.
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Each line of this table is formed from the preceeding one by inserting between consecutive elements their sum. The lines of this table, of which there are infinitely many, are numbered as indicated on the left.

Stern has proved a number of interesting facts concerning this series among which are the following:

1. The number of terms in the n-th line is 2^n+1, and their sum is 3^n+1.

2. The number of terms in the table down to and including the n-th line is $2^{n+1}+n$ and their sum is $\frac{1}{2}(3^{n+1}+1)+n$.

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[1] Journal für Mathematik, vol. 55, page 193.